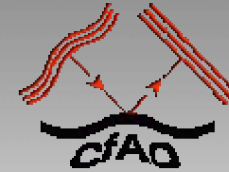


Center for Adaptive Optics
An NSF Science & Technology Center



Wave Optics

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University of California, Santa Cruz

CfAO Summer School on Adaptive Optics
August 6, 2006



Laboratory for Adaptive Optics

*UCO/Lick Observatory
University of California, Santa Cruz*

Course Objectives



- Gain a **physical understanding** of the wave and particle nature of light
- Acquire **immediately useful information** for use in the laboratory, for design studies, and theoretical development
- We will be **light on the math, heavy on the concepts**

Course Objectives (cont'd)



- Caveat
 - These note & lecture hopefully provide physical insight, but are not the definitive reference
- References
 - Anthony Siegman, **Lasers**, 1986 (Ch 16-20).
 - Web: **Eric Weisstein's World of Science**
scienceworld.wolfram.com/physics/topics/Optics.html
 - Joseph Goodman, **Introduction to Fourier Optics**, 1996
 - Max Born and Emil Wolf, **Principles of Optics**, 7th Ed., 2002.
 - George Reynolds, **The New physical optics notebook**, 1989.
 - Richard Feynman, **Lectures on Physics**, 1964.

Outline



- Light and photons
- Waves and interference
- Diffraction
- Fermat's principle, Marachal's condition, Lagrange Invariant
- Coherence
- Wave optics system modeling

The basics: what is light?



And God said

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J}$$

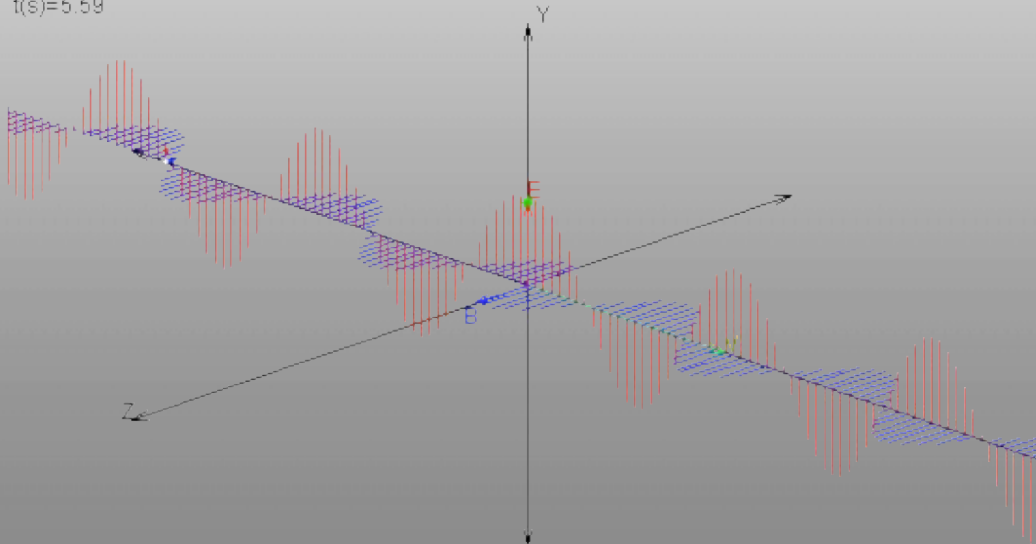
...and there was light!

Light as EM wave



- Light is an electromagnetic wave phenomenon
- Waves propagate in free space according to the Helmholtz equation
- We detect its presence because the EM field interacts with the electron $\mathbf{F} = q[\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]$

t(s)=5.59



Helmholtz Equation



- In free space

$$\nabla^2 E = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E$$

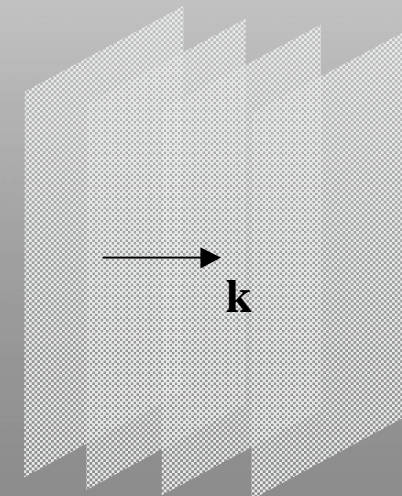
- Traveling waves

$$E(x, t) = E(0, t \pm x/c)$$

- Plane waves

Helmholtz Eqn.,
Fourier domain

$$E(\mathbf{x}, t) = \tilde{E}(\mathbf{k}) e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})}$$
$$k^2 E = (\omega/c)^2 E$$



Dispersion



- In free space

$$k = \omega/c$$

Dispersion relation is *linear*

$$k = 2\pi/\lambda$$

Wave number (\mathbf{k} is wave vector)

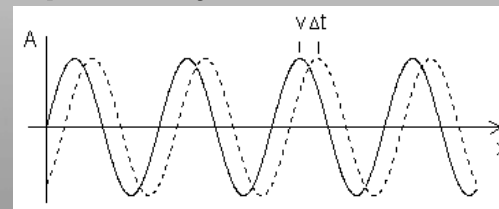
$$\omega = 2\pi\nu$$

Wave frequency, ν

- In a medium

- Plane waves have a *phase velocity*, and hence a wavelength, that depends on frequency

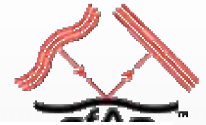
$$k(\omega) = \omega/v_{phase}$$



- The “slow down” factor is the *index of refraction*, $n(\omega)$

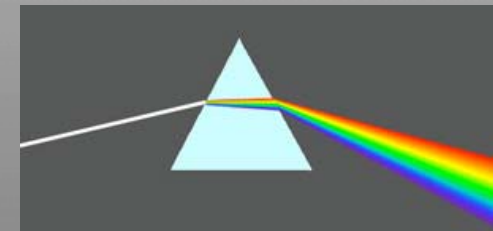
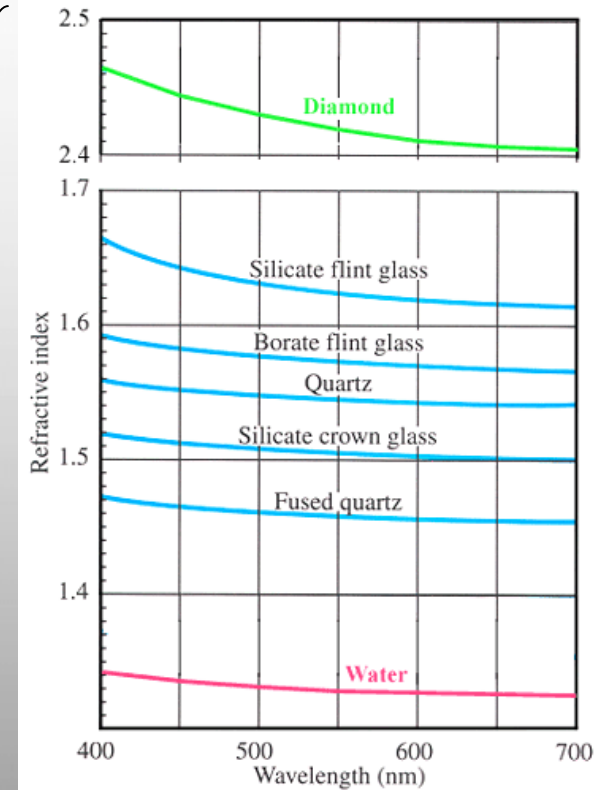
$$v_{phase} = c/n(\omega)$$

Some practical numbers

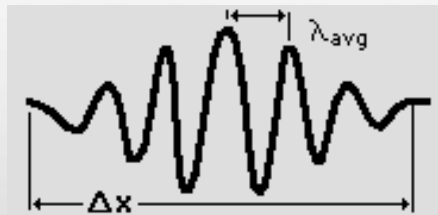


- Visible light
 - Wavelength $\lambda = 0.4 - 0.75 \mu\text{m}$
 - Velocity $c = 3 \times 10^8 \text{ m/s}$
 - Frequency $\nu = 6 \times 10^{14} \text{ Hz}$
- Index of refraction
 - Air: 1.00029
 - Glass: ~ 1.5
 - Water: 1.33

Dispersion



Wave Packets and Group Velocity



$$E(x, t) = \int \tilde{E}_k e^{i(\omega(k)t - kx)} dk$$

$$= e^{i(\omega_0 t - k_0 x)} \int \tilde{E}_k e^{i(\Delta\omega t - \Delta k x)} d\Delta k$$

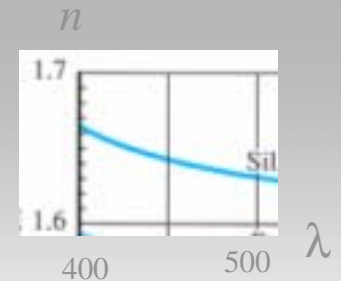
Plane wave

phase velocity $v_{\text{phase}} = \omega_0/k_0$

Envelope

group velocity $v_g = \Delta\omega/\Delta k$

Example:
Silica flint glass



$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\Delta(ck/n)}{\Delta k} = \frac{c}{n} - \frac{ck}{n^2} \frac{\Delta n}{\Delta k}$$

$$= \frac{c}{n} \left[1 - \frac{\Delta n/n}{\Delta k/k} \right] = \frac{c}{n} \left[1 + \frac{\Delta n/n}{\Delta\lambda/\lambda} \right]$$

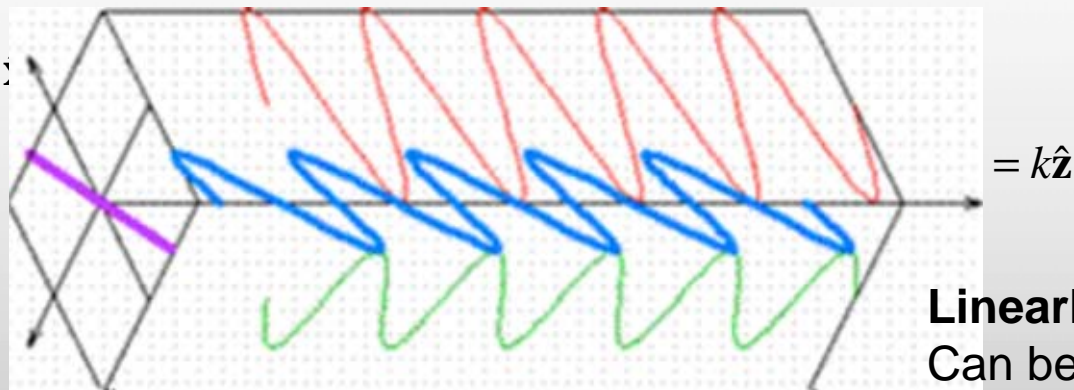
$$\lambda = 400\text{-}500\text{nm} \quad \frac{\Delta\lambda}{\lambda} = \frac{100}{450} = 0.222$$

$$n = 1.63\text{-}1.66 \quad \frac{\Delta n}{n} = \frac{-0.03}{1.645} = -0.207$$

$$v_g = 0.918 v_{\text{phase}}$$

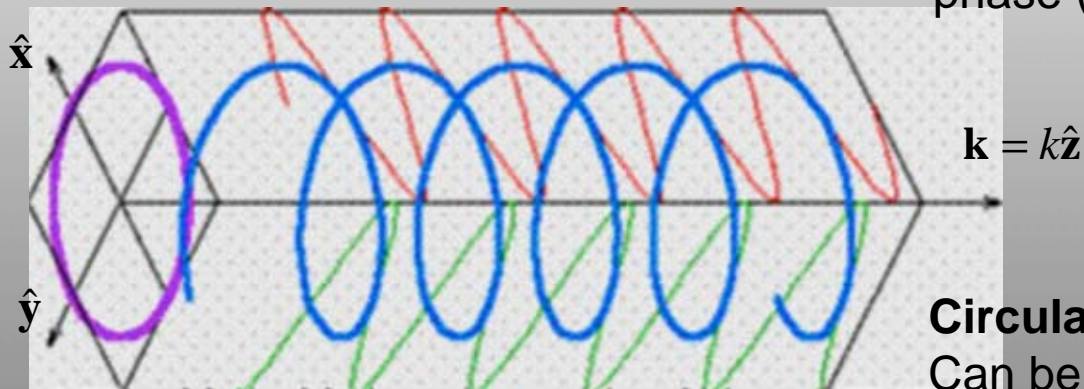
$$v_{\text{phase}} = \frac{c}{n} = 0.61c$$

Polarization



Linearly Polarized

Can be written as a sum of two linearly polarized waves, in phase (x and y components)



Circularly Polarized

Can be written as a sum of two linearly polarized waves, $\pi/2$ radians out of phase.

Polarization – Stokes parameters



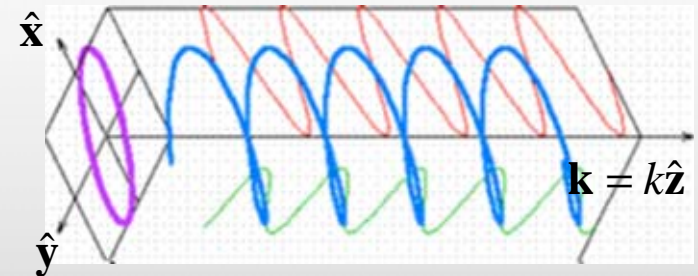
- 4 parameters of polarization
 - Total intensity
 - Rotation of ellipse
 - Ratio of major to minor axis of ellipse
 - Degree of polarization

$$\mathbf{E} = (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y)e^{i(\omega t - kz)} \quad E_y = |E_y|e^{i\delta}$$

$$I = \langle E_x E_x^* + E_y E_y^* \rangle \quad Q = \langle E_x E_x^* - E_y E_y^* \rangle$$

$$U = \langle E_x E_y^* + E_y E_x^* \rangle \quad V = i \langle E_x E_y^* - E_y E_x^* \rangle$$

- Waves can be a sum of mixed polarization waves (“quasi-monochromatic”)



- Degree of polarization

$$\Pi = \sqrt{Q^2 + U^2 + V^2} / I$$

- Degree of linear polarization

$$\Pi_L = \sqrt{Q^2 + U^2} / I$$

- Degree of circular polarization

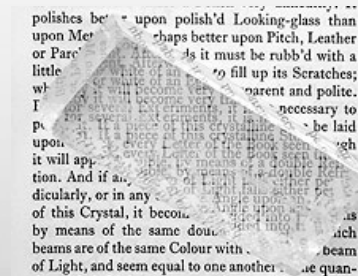
$$\Pi_C = V / I$$

<http://scienceworld.wolfram.com/physics/StokesParameters.html>

Birefringence



- Medium can have different index of refraction for each component of polarization
- Polarization splitter (Wallaston Prism)
- Waveplates
 - Linear to circular polarization ($\lambda/4$ plate)
 - Rotate linear polarization ($\lambda/2$ plate)



Optical path – Fermat's principle

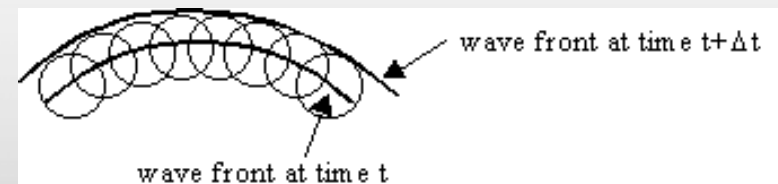


- Huygens' wavelets
- Optical distance to radiator

$$\Delta x = c\Delta t/n$$

$$OPD = \int n dx$$

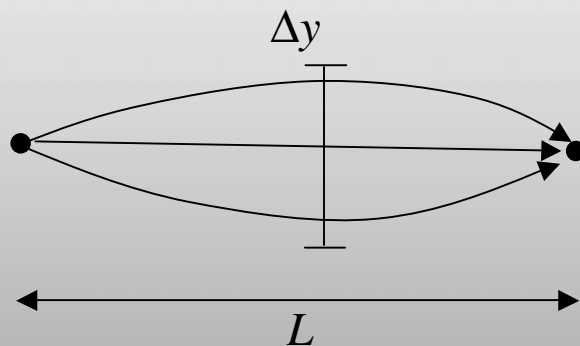
$$\phi = \omega\Delta t = c \times OPD$$



- Wavefronts are iso-OPD surfaces
- Light ray paths are paths of least* time (least* OPD)

*in a local minimum sense

Why doesn't light prefer other paths?



- Waves arriving in phase add
- Waves arriving out of phase cancel

$$OPD_0 = L$$

$$OPD_{\Delta} = L + \frac{1}{2} \frac{\Delta y^2}{L}$$

$$OPD_{\Delta} - OPD_0 < \lambda/2 \Rightarrow \Delta y < \sqrt{\lambda L}$$

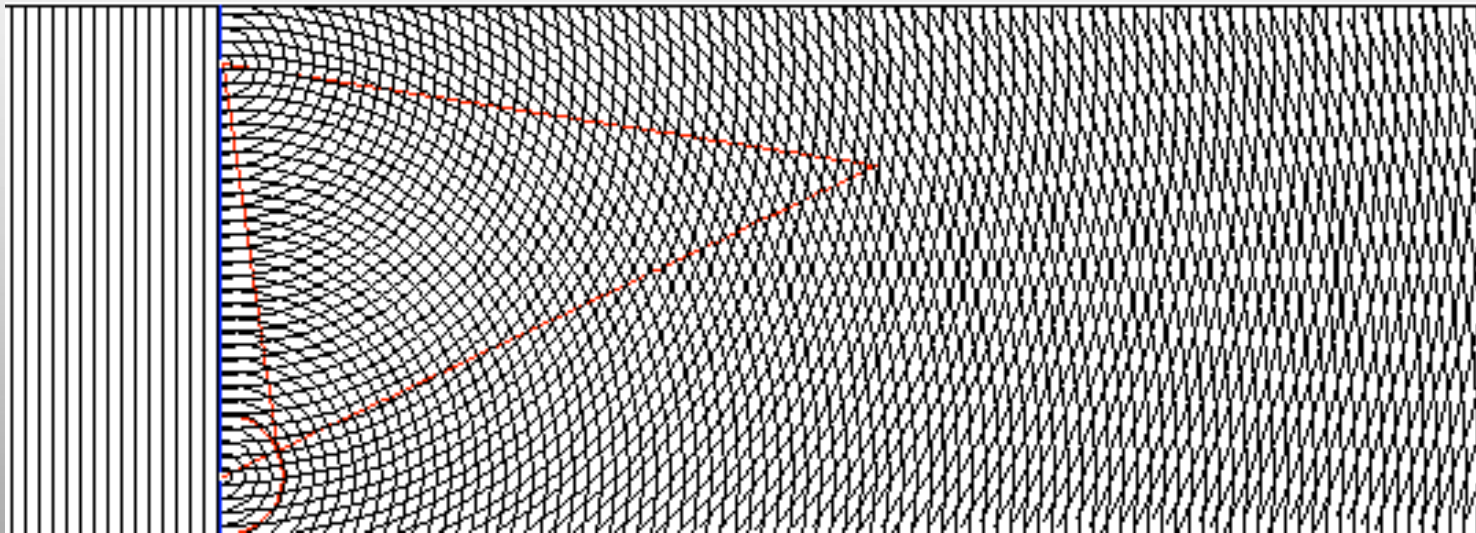
Fresnel zone

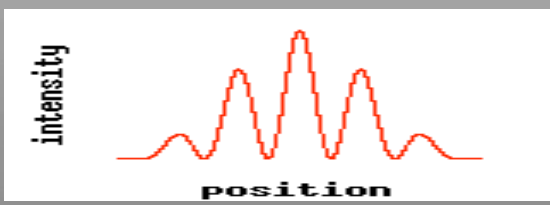
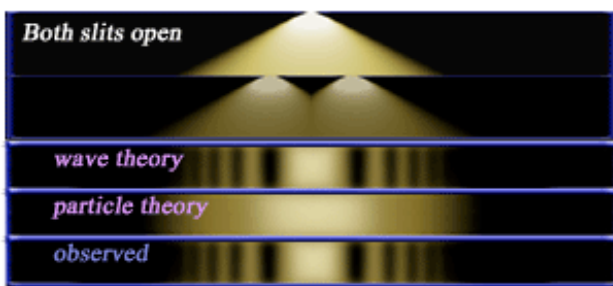
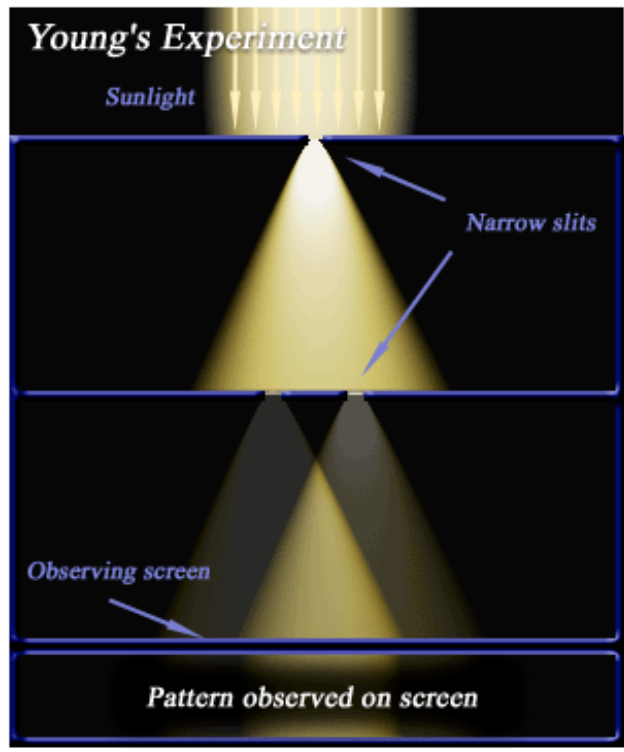
Light as particles



- Light originates as photons each emitted by the oscillation of a single atom.
- Light travels as a wave via all possible paths (paths of 'least time').
- Light, when detected, is realized as single-photon events distributed according to the intensity of the wave.

Detecting photons – Young's double slit experiment

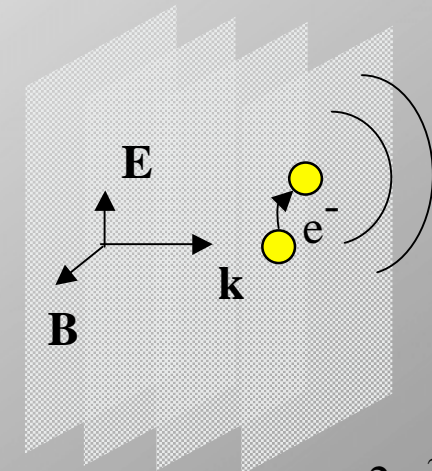




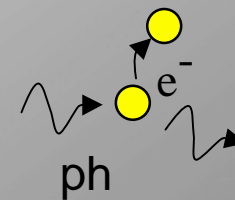
Compton scattering



- Photons interact only with charged particles (electrons, in ordinary life)
- Classical: The EM wave exerts force on the electron. The electron moves and thus emits an EM wave.
- Quantum: A photon collides with an electron and exchanges energy and momentum with it. The photon scatters with a different wavelength.



$$m\ddot{\mathbf{x}} = q[\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}] - \frac{2}{3} \frac{e^2}{c^3} \ddot{\mathbf{x}}$$



$$\lambda' - \lambda = \frac{2h}{mc} \sin^2 \frac{\theta}{2}$$

Light has energy and momentum



- Classical: electromagnetic field energy and momentum density

$$E_{EM} = \frac{1}{8\pi} \int (E^2 + B^2) dV$$

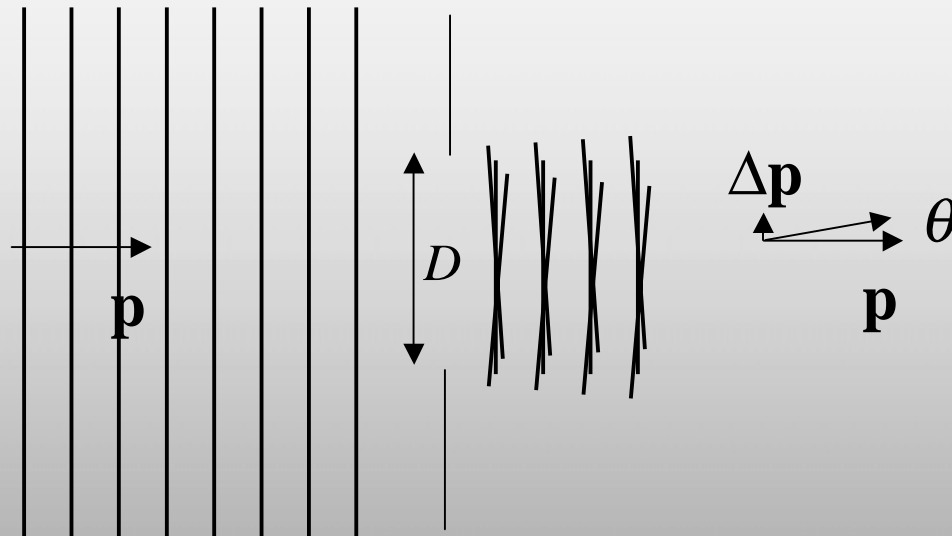
$$\mathbf{p}_{EM} = \frac{E_{EM}}{c} \hat{\mathbf{k}} = \frac{1}{4\pi c} \int (\mathbf{E} \times \mathbf{B}) dV$$

- Quantum: photon energy and momentum

$$E_{ph} = h\nu$$

$$\mathbf{p}_{ph} = \frac{E_{ph}}{c} \hat{\mathbf{k}} = h\mathbf{k}$$

Diffraction as a particle phenomenon



Uncertainty principle

$$\Delta x \Delta p \cong h$$

Photon momentum

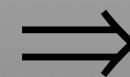
$$p = h/\lambda$$

$$\Delta \mathbf{x} = \bullet$$

$$\Delta \mathbf{p} = 0$$

$$\Delta x = D$$

$$\Delta p = h/D$$



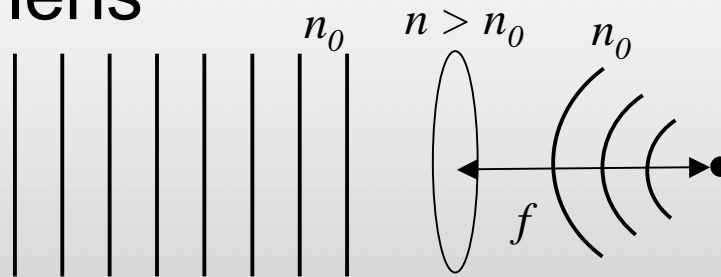
Law of diffraction

$$\theta \cong \Delta p / p = \lambda / D$$

Ideal lens



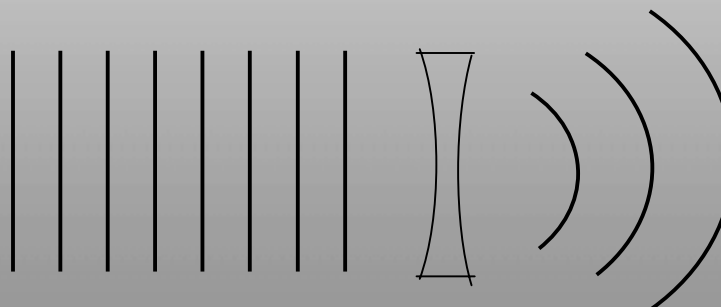
- Positive lens



Plane Wave

Converging spherical Wave

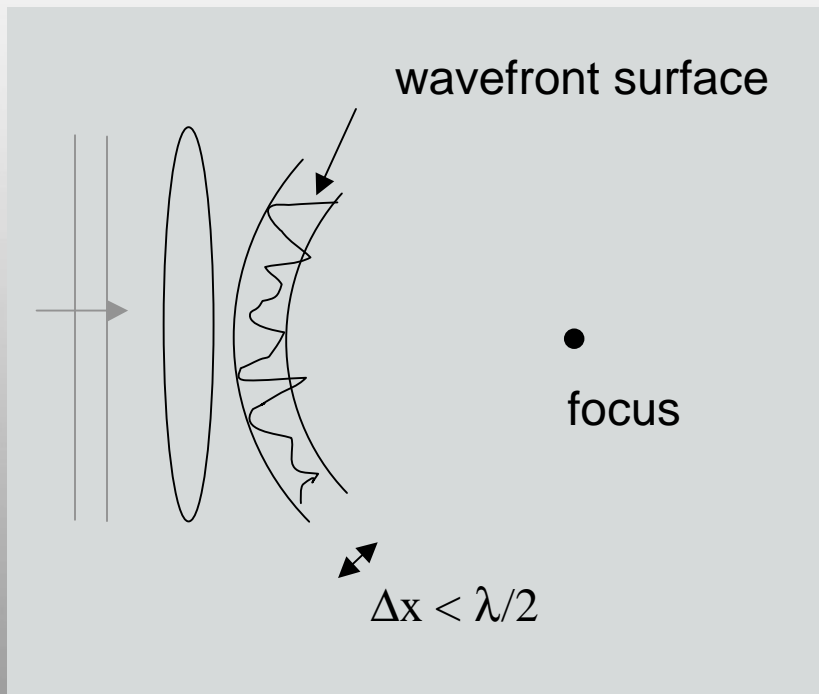
- Negative lens



Plane Wave

Diverging spherical Wave

Marechal's condition

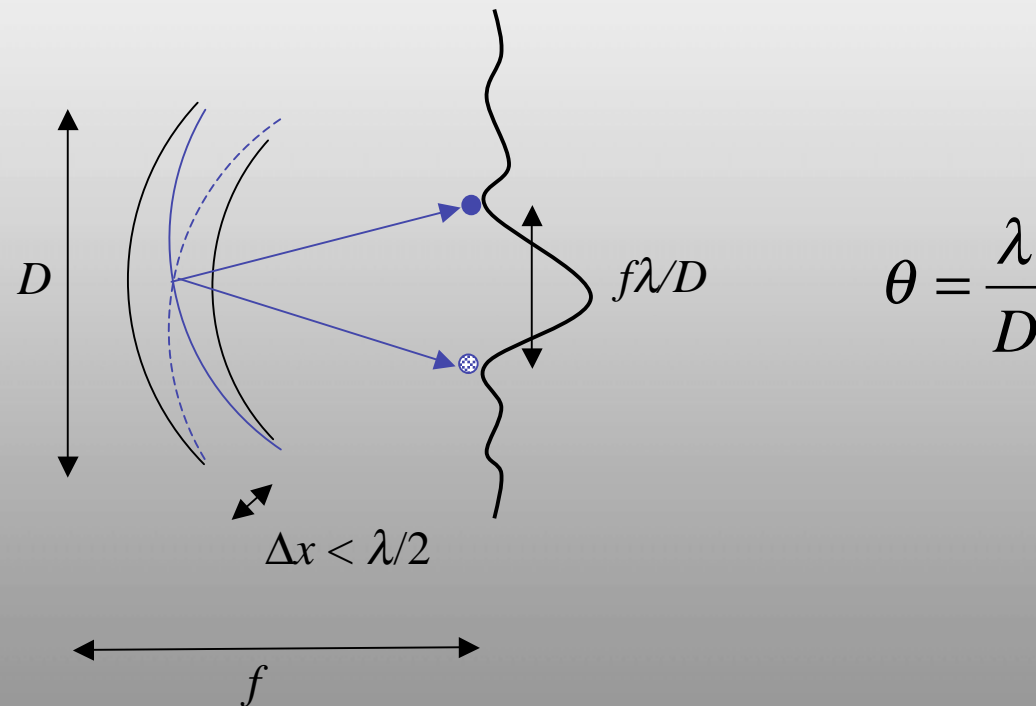


- If wavefront phase is contained within confocal spheres $\lambda/2$ apart everywhere where the intensity is significant
- The waves will add up at the focus
- Consequence of Fermat's principle

Diffraction angle



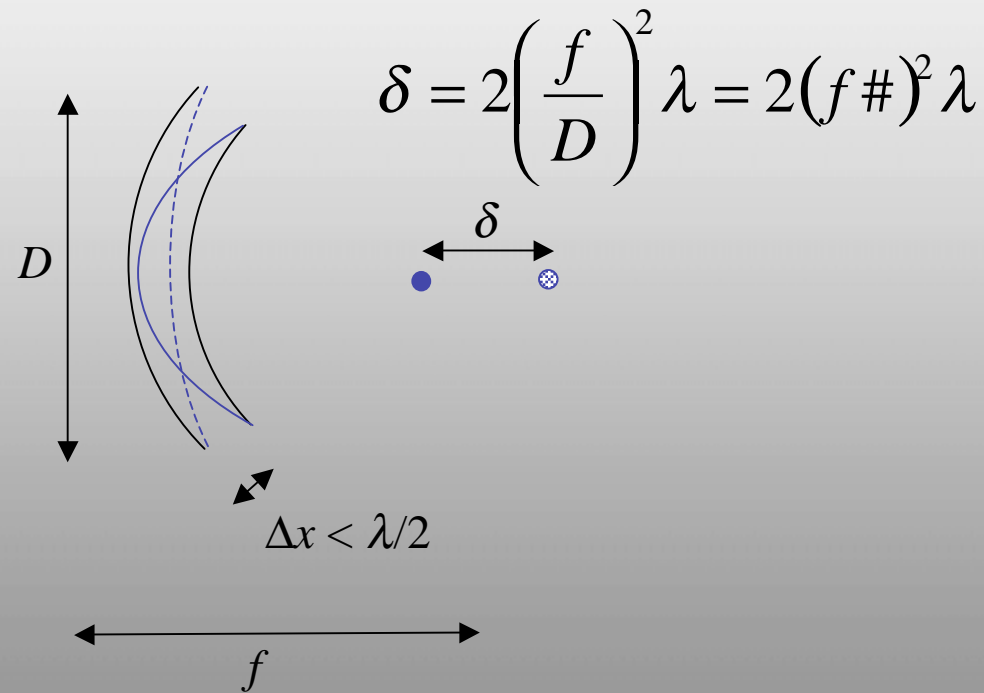
- Tip/Tilt allowed by Marechal's condition



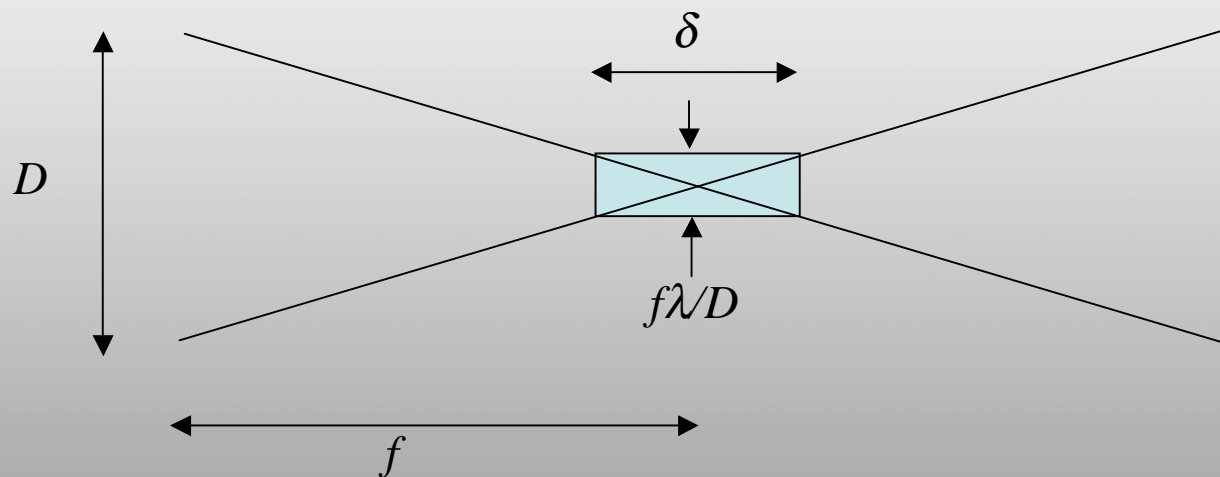
Depth of Focus



- Defocus allowed by Marechal's condition

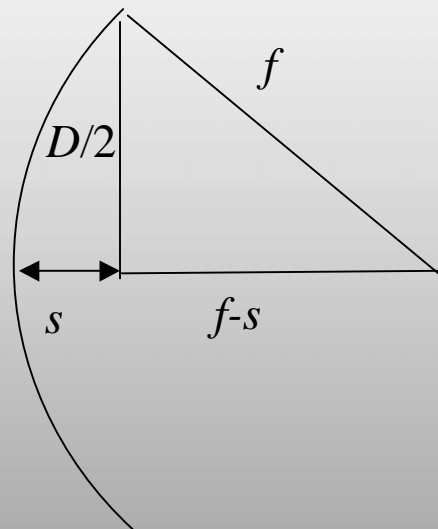


Another derivation of depth of focus



$$\frac{f}{D} = \frac{\delta/2}{f \lambda/D} \Rightarrow \delta = 2 \left(\frac{f}{D} \right)^2 \lambda$$

Wavefront sag



$$f^2 = (f - s)^2 + (D/2)^2$$

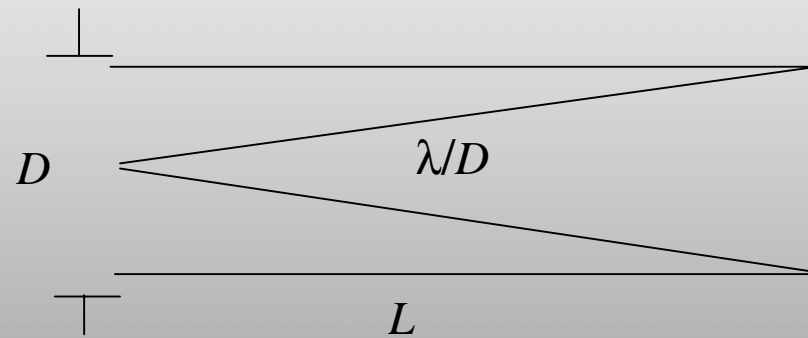
$$\cong f^2 - 2fs + D^2/4$$

$$s = \frac{D^2}{8f}$$

Rayleigh range



- Distance where diffraction overcomes paraxial beam propagation



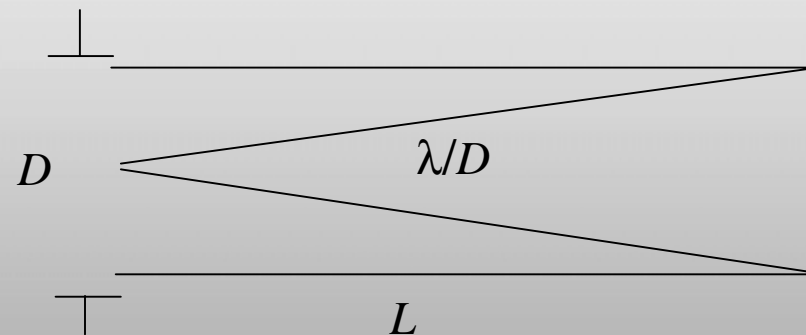
$$\frac{L\lambda}{D} = D \Rightarrow L = \frac{D^2}{\lambda}$$

- Also: wavefront sag is less than half a wave

Fresnel number



- Number of Fresnel zones across the beam diameter



$$N = \frac{D}{L\lambda/D} = \frac{D^2}{L\lambda}$$

Paraxial beams



- Helmholtz equation

$$\nabla^2 E = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E$$

$$E(x, y, z, t) = u(x, y, z) e^{i(-kz)} e^{i\omega t}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - 2ik \frac{\partial u}{\partial z} - k^2 u = -\frac{\omega^2}{c^2} u, \quad k = \frac{\omega}{c}$$

- Paraxial approximation

$$\left| \frac{\partial^2 u}{\partial z^2} \right| \ll \left| 2k \frac{\partial u}{\partial z} \right| \text{ or } \left| \frac{\partial^2 u}{\partial x^2} \right| \text{ or } \left| \frac{\partial^2 u}{\partial y^2} \right|$$

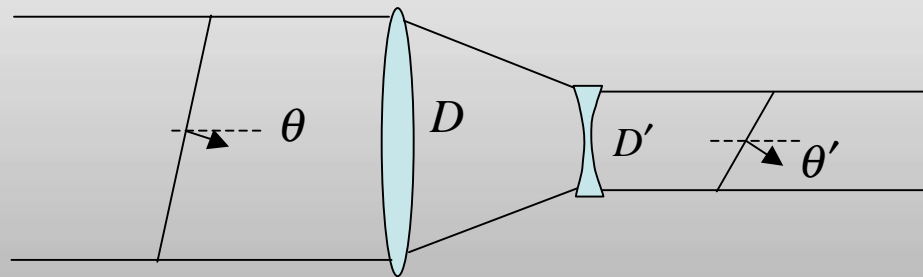
- Paraxial wave equation

$$\nabla_t^2 u - 2ik \frac{\partial u}{\partial z} = 0$$

Lagrange invariant



- If the beam diameter is condensed, the angles increase proportionally



$$\Xi = \theta D = \theta' D'$$

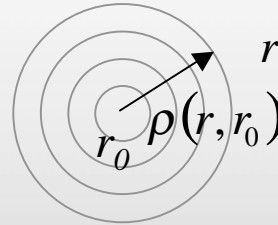
- Conservation of energy
 - Flux is proportional to Ξ^2

Huygens' Integral

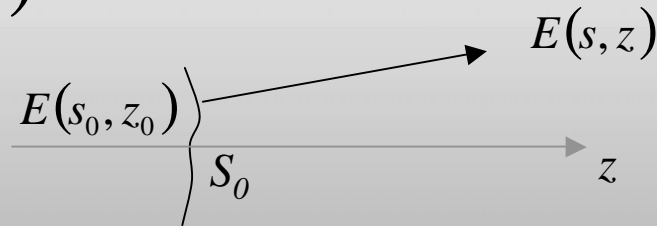


- Wavelet

$$E(r; r_0) = \frac{e^{-ik\rho(r, r_0)}}{\rho(r, r_0)}$$



- Huygens' integral



$$E(s, z) = \frac{i}{\lambda} \iint_{S_0} E(s_0, z_0) \frac{e^{-ik\rho(r, r_0)}}{\rho(r, r_0)} \cos \theta(r, r_0) dS_0$$

Fresnel approximation



- Spherical wavelet approximated as paraboloid
- Useful for computer numerical wave-optic propagation (it's a convolution)

$$u(x, y, z) = \frac{i}{L\lambda} \iint u_0(x_0, y_0, z_0) \exp\left[-i\pi \frac{(x-x_0)^2 + (y-y_0)^2}{L\lambda}\right] dx_0 dy_0$$

- Fourier domain: plane wave components

$$\mathbf{k} = (k_x, k_y, k_z)$$

$$k_{\perp} = k(\sin\theta_x, \sin\theta_y)$$

$$k \cong k_z$$

$$\tilde{u}(k_{\perp}, L) = \tilde{u}(k_{\perp}, 0) \times \exp[i\lambda k_{\perp}^2 L] \times e^{-ikL}$$

Fresnel propagation in IDL

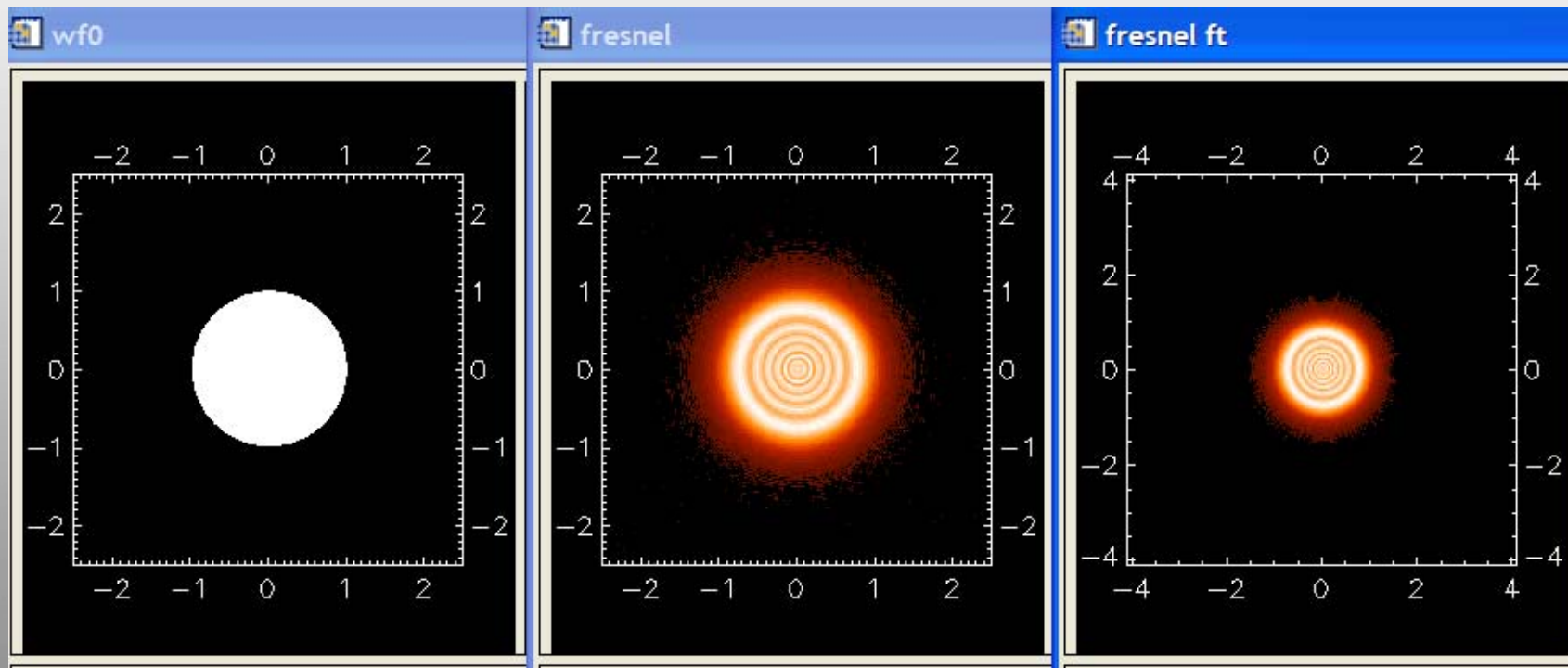


```
;  
; fresnel.pro - fresnel propagation  
;  
; Implement's Fresnel's approximation to Huygens' integ  
; for a complex wavefront propagating paraxially from a  
; z=0 to a plane at z=L.  
; Refernce: Siegman, Ch 16, eqn (79)  
; translated from fresnel.vm  
;  
; USAGE:  
; wfL = fresnel(wf0,du,L,lambda)  
;  
; INPUTS:  
; wf0 - complex wavefront at a given optical plane  
; du - spacing on the wavefront grid, in meters  
; L - real length of propagation, in meters  
; lambda - real wavelength of light, in meters  
;  
; OUTPUT:  
; wfL - complex wavefront at distance L  
;  
; ALGORITHM:  
; ~wfL = ~wf0 * exp{ + i k_perp^2 * L / 2*k }  
; where ~ indicates Fourier transform  
;-
```

```
function fresnel,wf0,du,l,lambda  
  k = 2*!pi/lambda  
  n = (size(wf0))(1)  
  df = 1./(n*du)  
  dk = 2.*!pi*df  
  fwf0 = shift(fft(shift(wf0,n/2,n/2)),n/2,n/2)  
  k0 = -(n/2)*dk  
  kf = (n/2-1)*dk  
  r = findgen(n)*dk + k0  
  kx = transpose(r) ## make_array(n,1,/float,value=1)  
  ky = transpose(kx)  
  kperp2 = kx*kx + ky*ky  
  propPhase = kperp2*L/(2*k)  
  propMag = 0*kperp2 + 1  
  i = complex(0,1)  
  prop = propMag*Exp(i*propPhase)  
  fwfl = fwf0*prop  
  wfL = shift(fft(shift(fwfl,n/2,n/2),/inverse),n/2,n/2)  
  return,wfL  
end
```

Plane wave propagation from a circular aperture

$L = 2\%$ of Rayleigh range



An alternative computational formula



- Fresnel approx to Huygens' integral (again)

$$u(x, y, z) = \frac{i}{L\lambda} \iint u_0(x_0, y_0, z_0) \exp\left[-i\pi \frac{(x-x_0)^2 + (y-y_0)^2}{L\lambda}\right] dx_0 dy_0$$

- Alternative Fourier transform formulation for numerical propagation

$$u(x, L) = \exp\left(\frac{i\pi x^2}{L\lambda}\right) \sqrt{\frac{i}{L\lambda}} \iint u'(x_0, 0) \times \exp\left[i\frac{2\pi}{L\lambda} xx_0\right] dx_0$$

$$u'(x_0, 0) = u_0(x_0, z_0) \exp\left(-i\frac{\pi x_0^2}{L\lambda}\right)$$

Far field



- Each tilted plane wave produces a point image, at angle θ
- Sum of plane waves = sum of points = Fourier transform

$$u(\theta_x, \theta_y, \bullet) \propto \iint u_0(x, y, 0) \exp(ik\hat{e} \cdot \mathbf{x}) dx dy$$

- This is just a variation of the alternative form, with $L \rightarrow \bullet$

When to use each method

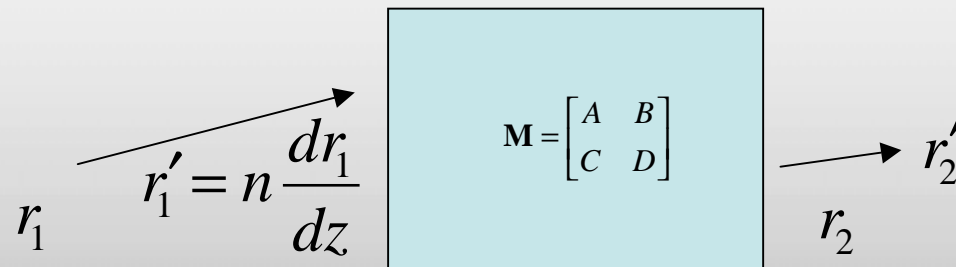


- Fresnel approximation
 - $L < \sim 10\%$ of Rayleigh range
- Fresnel approximation alternative form
 - $L > \sim 10\%$ of Rayleigh range
- Far field
 - $L = \bullet$


ABCD ray optics



- Optics systems as 2x2 linear ray transformations

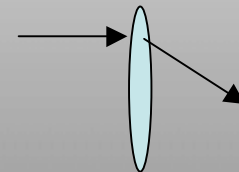


- Free space propagation

$$\begin{bmatrix} 1 & L/n \\ 0 & 1 \end{bmatrix}$$


- Thin lens

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$



- Curved mirror
- Dielectric interface
- Transversely graded index
- ...

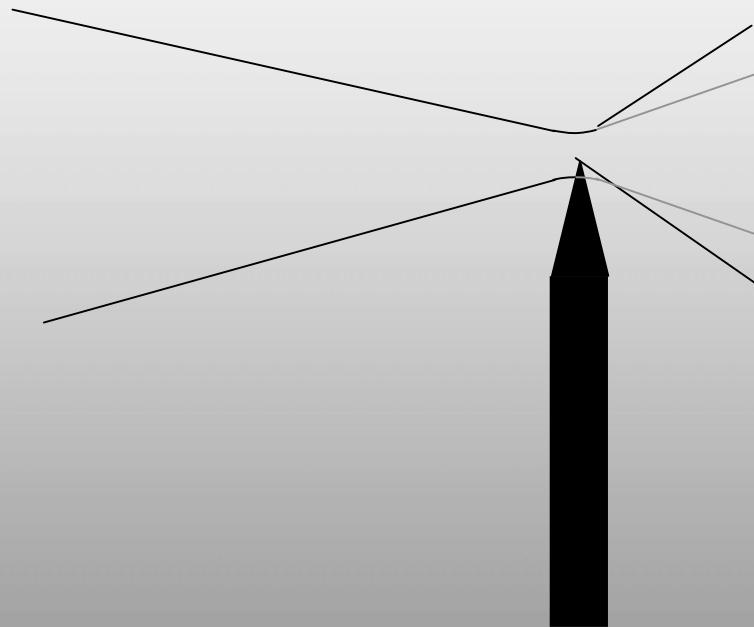
Huygens' integral (Fresnel approximation) through ABCD system



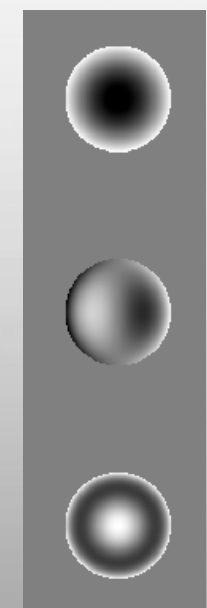
$$u_2(x_2) = e^{-ikL_0} \iint K(x_2, x_1) u_1(x_1) dx_1$$

$$K(x_2, x_1) = \sqrt{\frac{i}{B\lambda_0}} \exp\left[-i \frac{\pi}{B\lambda_0} (Ax_1^2 - 2x_1x_2 + Dx_2^2)\right]$$

Example: Knife Edge Test



Example: Knife Edge Test

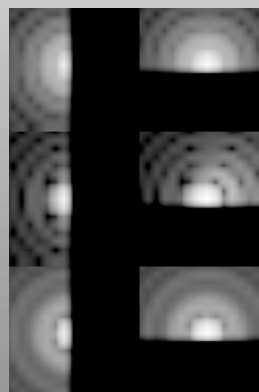
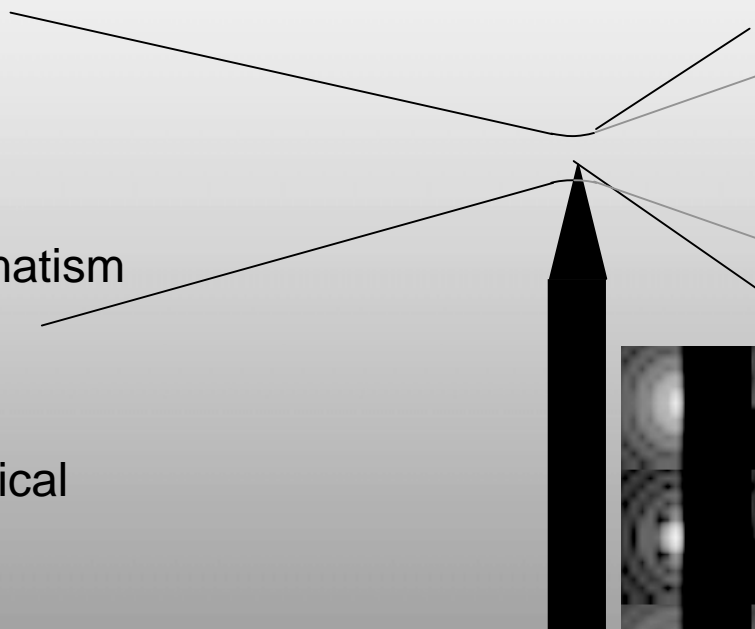


Focus

Astigmatism

Spherical

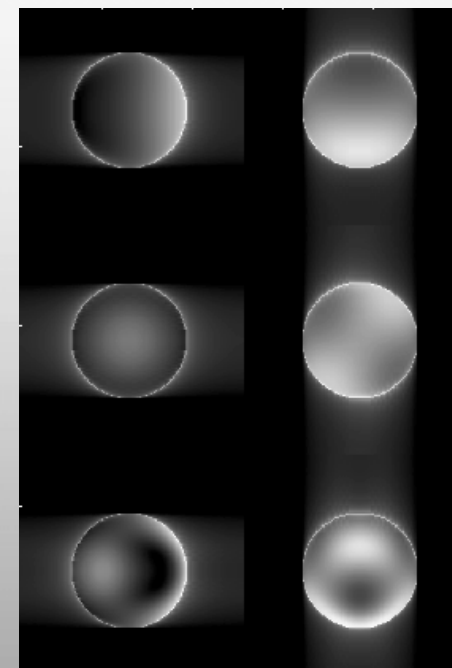
Phase
Aberration



X

Y

At focal plane

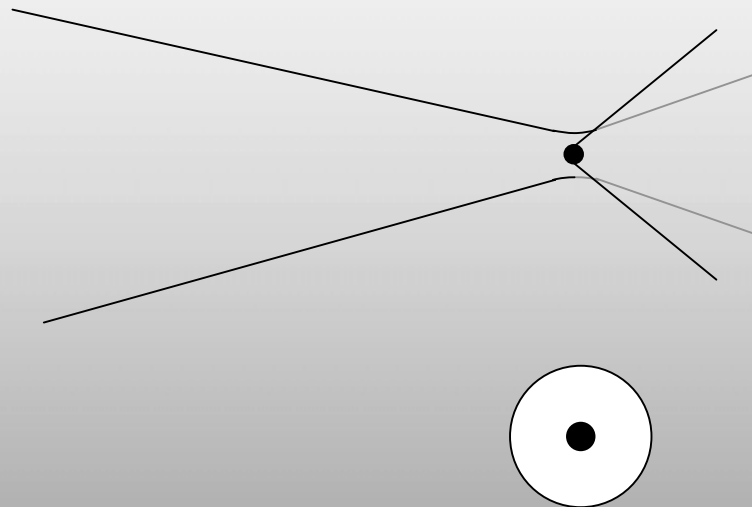


X

Y

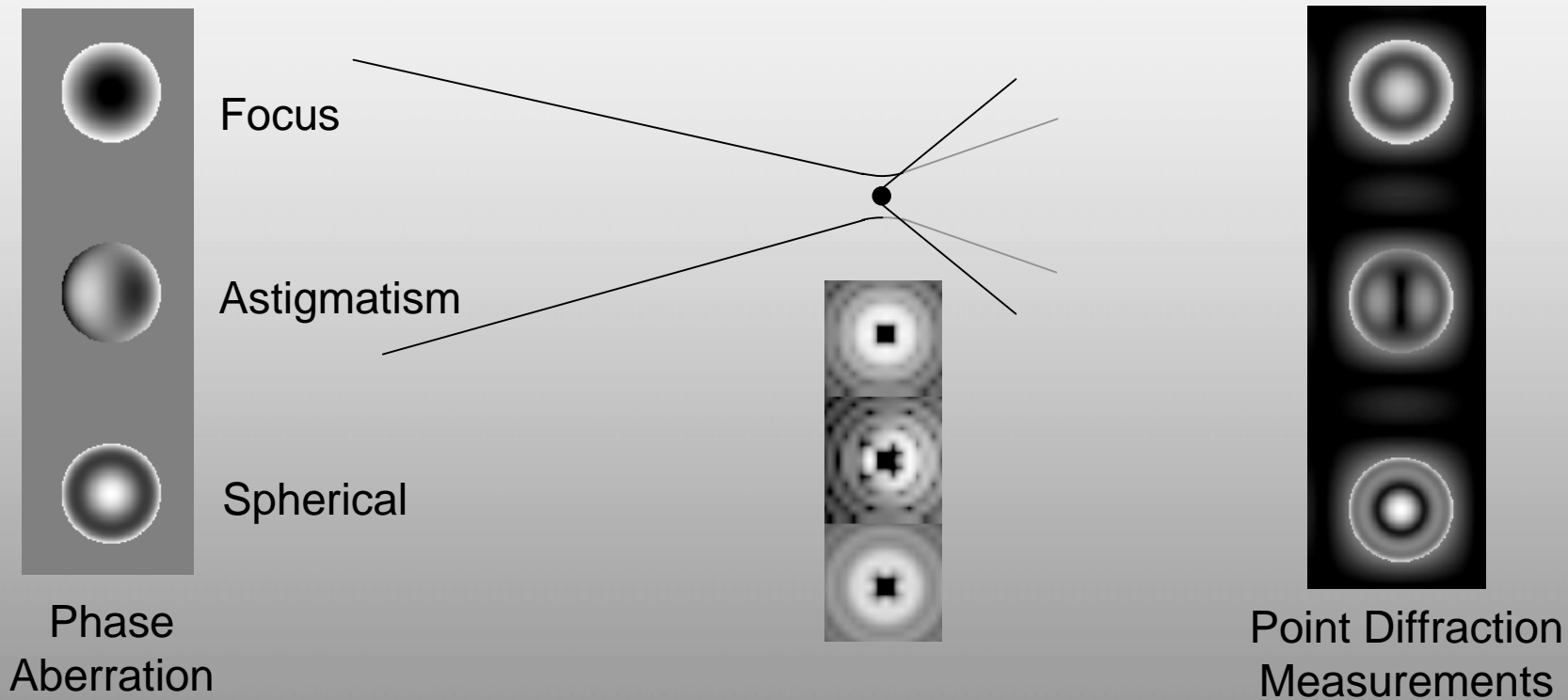
Knife Edge
Measurements

Example: Point Diffraction

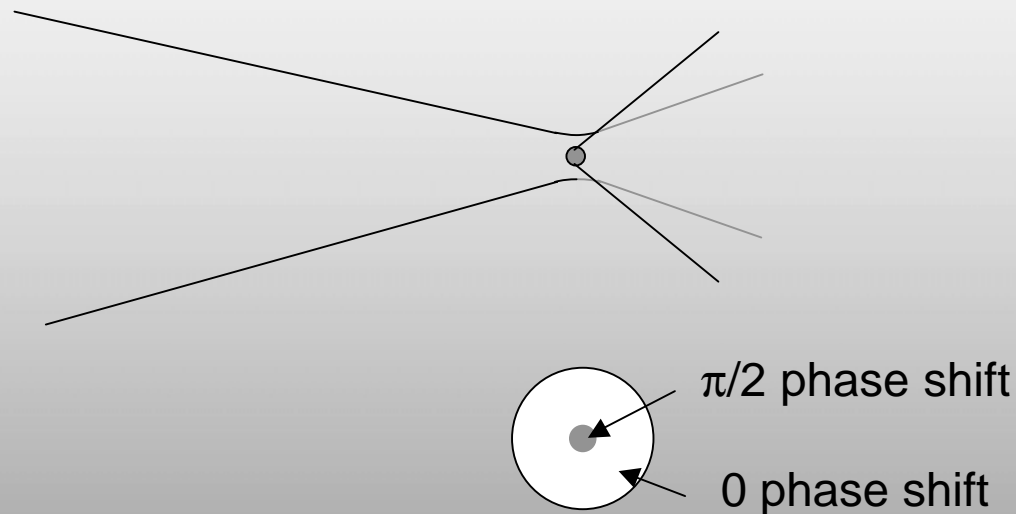


- Focal Plane Block at center of beam approximately $1/2$ diffraction limit

Example: Point Diffraction

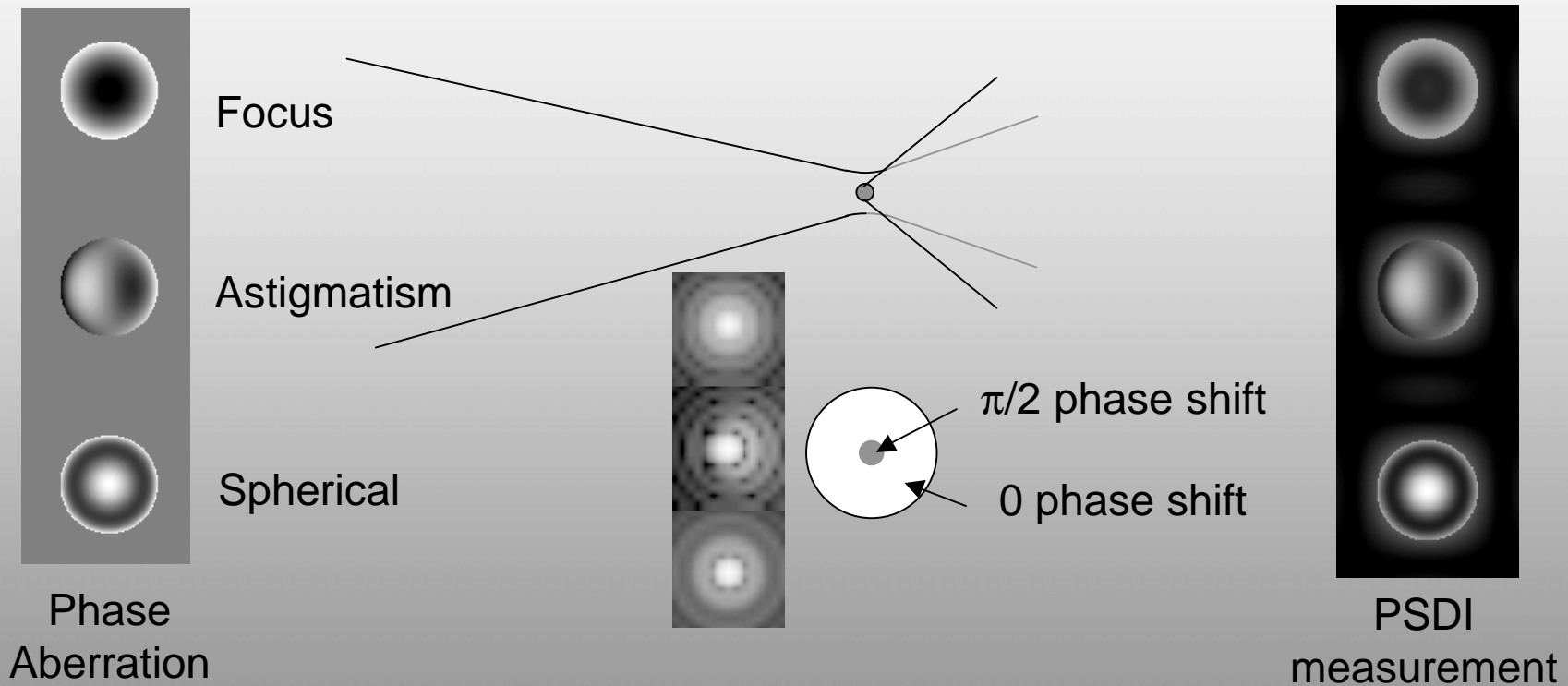


Example: Phase Shifting Point Diffraction Interferometer



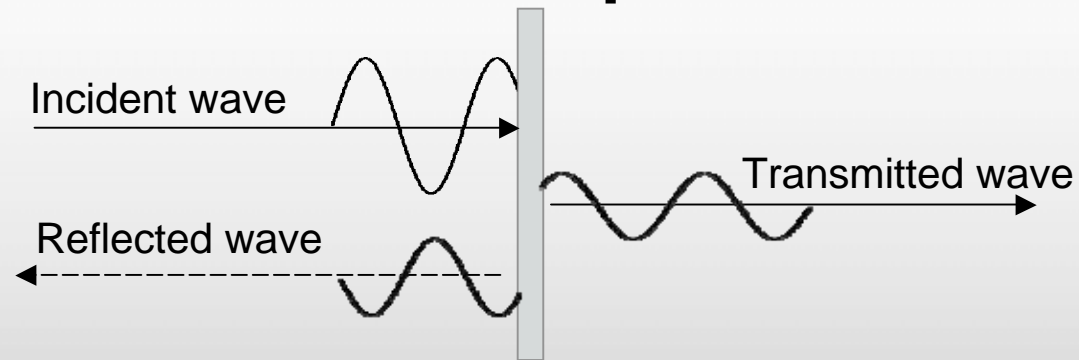
- Focal Plane Pinhole and $\pi/2$ wave phase shift at center of beam approximately $1/2$ diffraction limit

Example: Phase Shifting Point Diffraction Interferometer

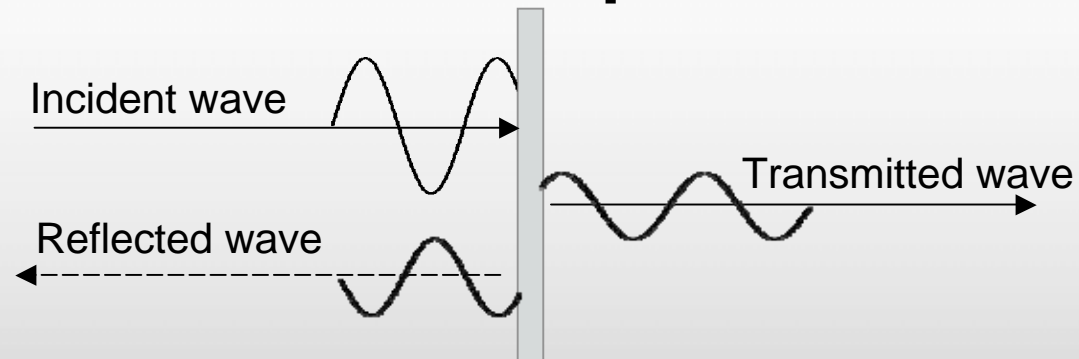


- Focal Plane Pinhole and $\pi/2$ wave phase shift at center of beam approximately $1/2$ diffraction limit

Beam Splitter



Beam Splitter



At the interface:

- EM wave magnitudes must match
- Energy must balance

$$E_i + E_r = E_t$$

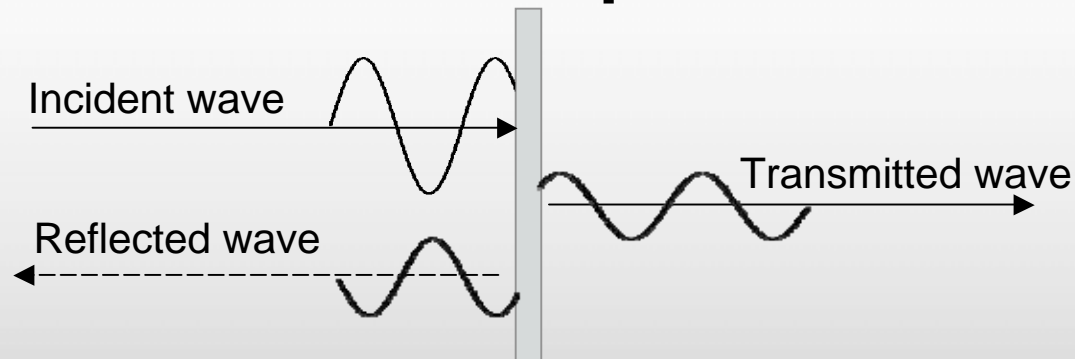
$$|E_i|^2 = |E_r|^2 + |E_t|^2$$

$$|E_t|^2 = T|E_i|^2$$

} E complex, ->
3 equations in
4 unknowns

Free variable is Transmissivity
of the beam splitter

Beam Splitter



At the interface:

- EM wave magnitudes must match
- Energy must balance

$$E_i + E_r = E_t$$

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Free variable is Transmissivity
of the beam splitter

$$|E_t|^2 = T|E_i|^2$$

a little math, and... $\frac{|E_t|}{|E_i|} = \cos(\phi_t)$ $\frac{|E_r|}{|E_i|} = -\cos(\phi_r) \Rightarrow \phi_t - \phi_r = -\pi/2$

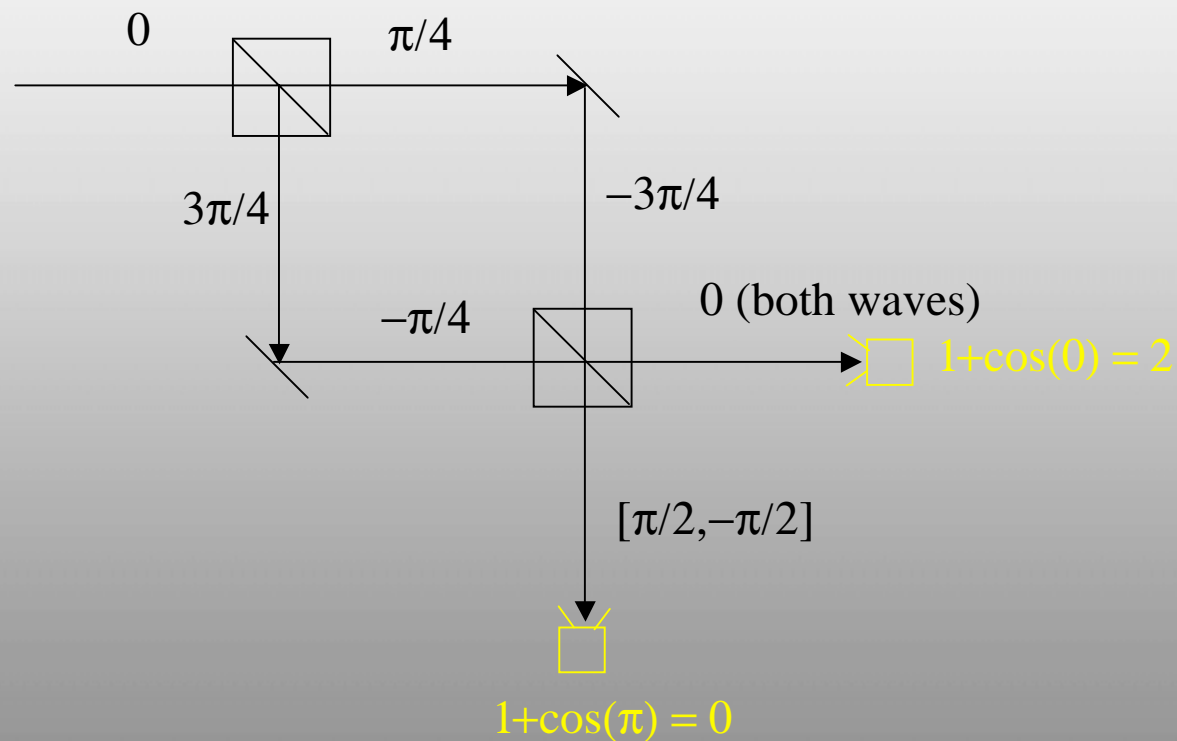
The transmitted and reflected waves differ by 90°

- Special case: 50/50 beamsplitter
- Special case: mirror

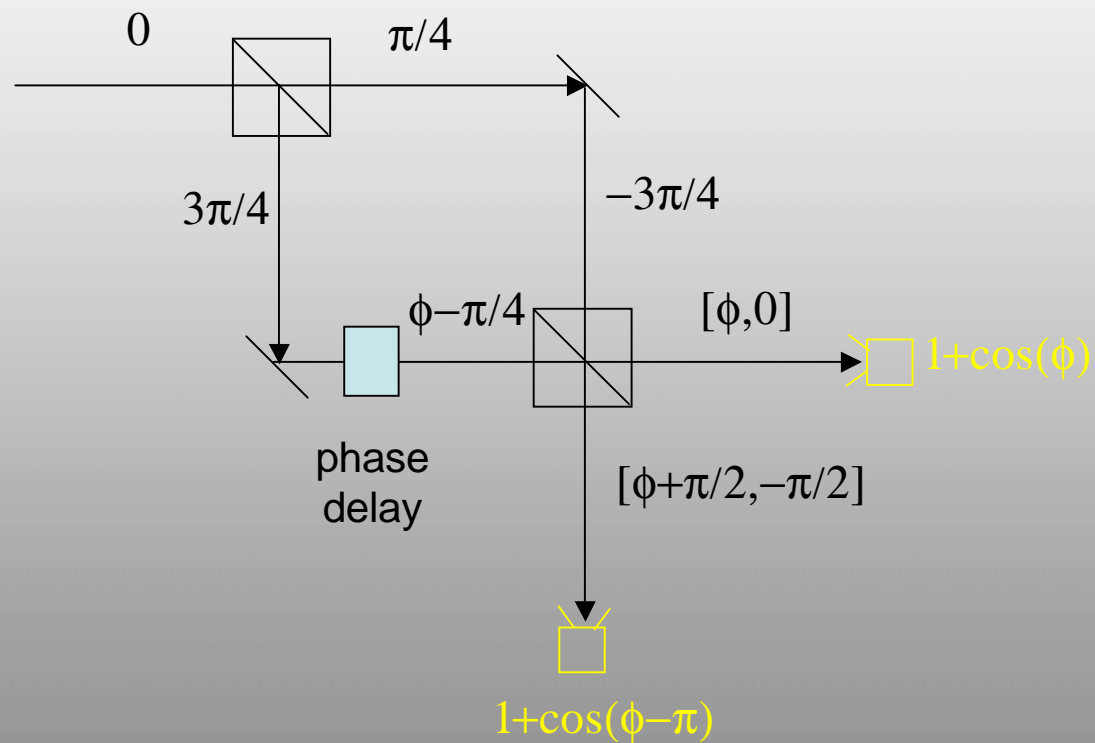
$$T = 0.5 \quad \phi_t = \pi/4 \quad \phi_r = 3\pi/4$$

$$T = 0 \quad \phi_t = \pi/2 \quad \phi_r = \pi$$

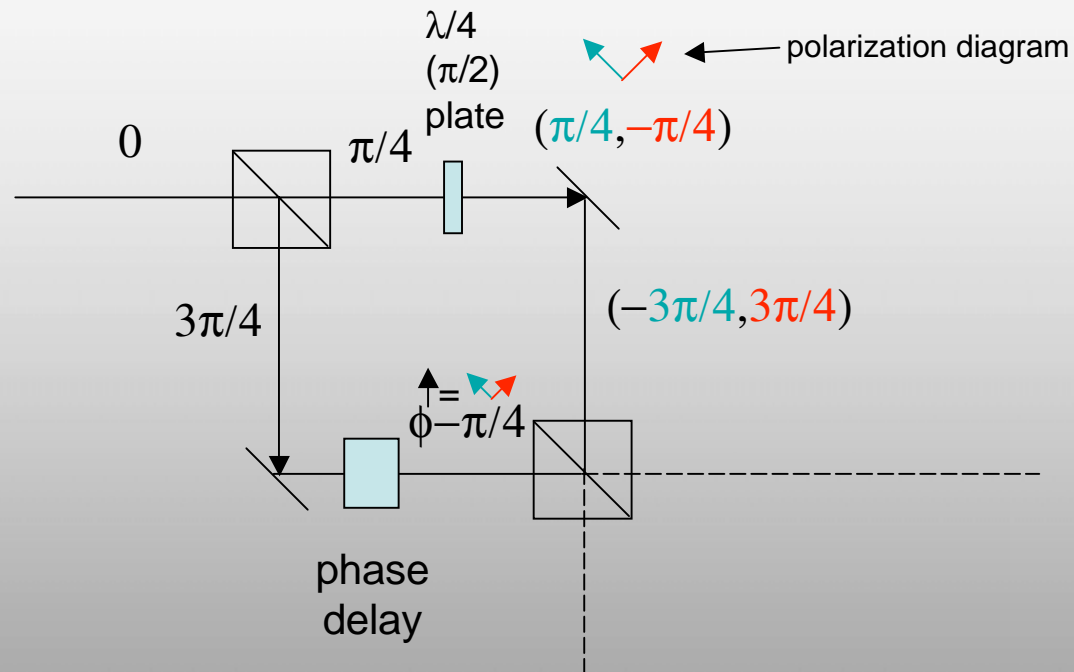
Beam Combiner (Interferometer)



Interferometric Testing



Quadrature Phase Interferometer



Quadrature Phase Interferometer

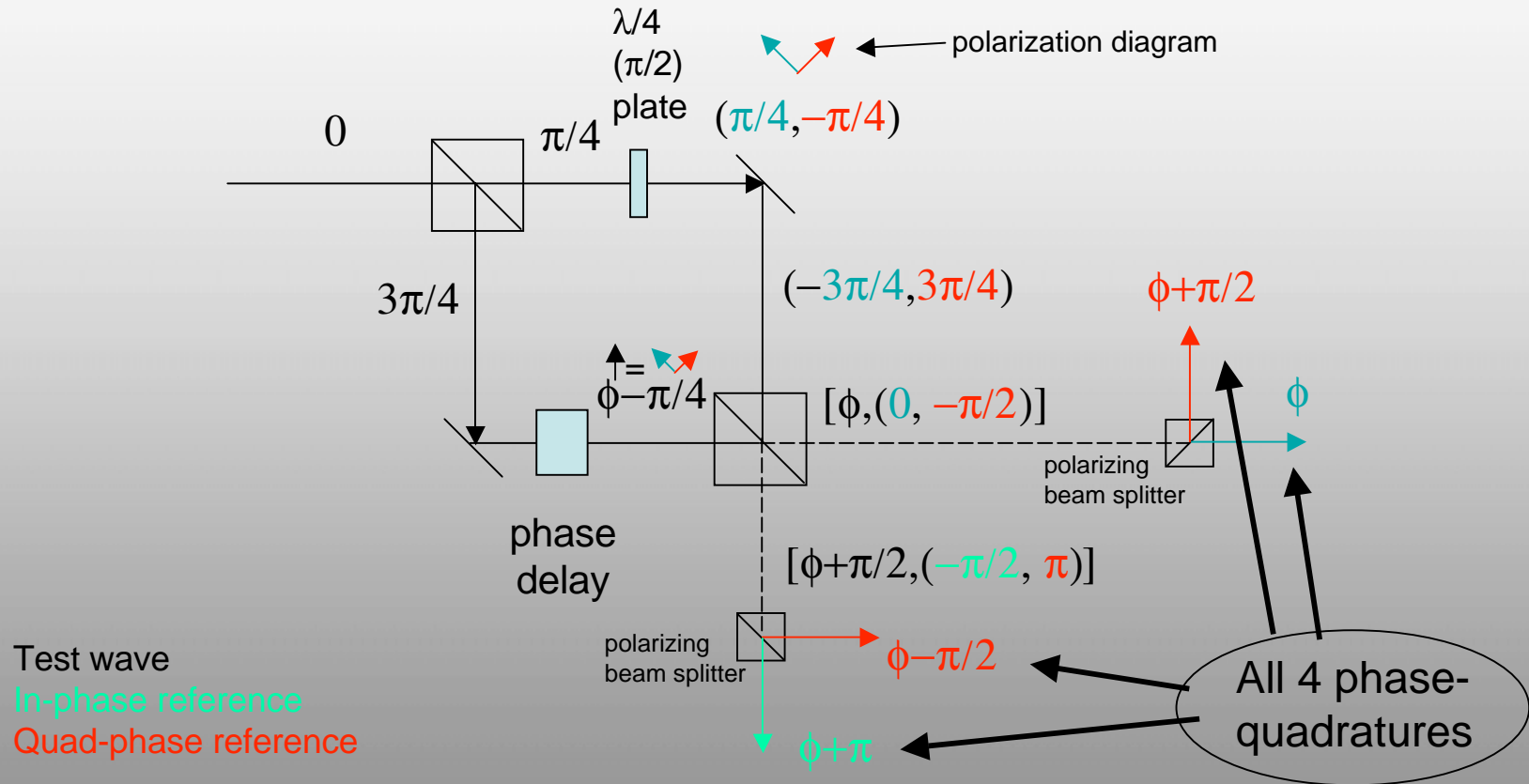


Image formation using Fourier optics

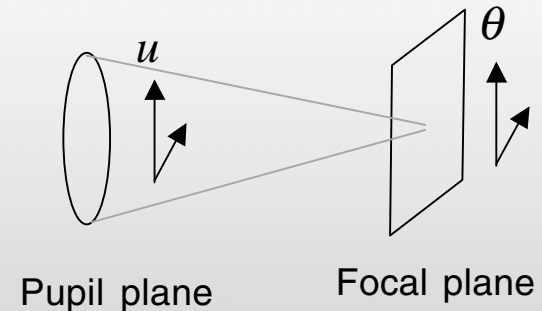


- Field at the pupil plane

$$E(u) = A(u)e^{i\phi(u)}$$

- Field at the focal plane

$$F(\theta) = \frac{1}{\lambda} \int E(u) e^{i2\pi u \cdot \theta / \lambda} du$$



- Average PSF at the focal plane

$$\langle PSF(\theta) \rangle = \langle |F(\theta)|^2 \rangle$$

$$\frac{1}{\lambda^2} \iint A(u)A(u+r) \langle e^{i(\phi(u)-\phi(u+r))} \rangle e^{-i2\pi r \cdot \theta / \lambda} dudr$$

Image formation (continued)



$$\langle A(u)A(u+r) \rangle_A = B_0(r)$$

Optical Transfer
Function (OTF) of the
Pupil

$$\langle e^{i(\phi(u)-\phi(u+r))} \rangle = e^{\frac{1}{2}\langle (\phi(u)-\phi(u+r))^2 \rangle} = e^{-\frac{1}{2}D_\phi(r)}$$

OTF of the Atmosphere

$$\langle PSF(\theta) \rangle = \frac{A_0}{\lambda^2} \int B_0(r)B_A(r)e^{-i2\pi r \cdot \theta/\lambda} dr$$

PSF = F (OTF)

$$\underline{\text{Strehl}}: PSF(0) / PSF(0) |_{\phi=0}$$

Optical and Modulation Transfer Functions



- Optical transfer function (OTF) – how a sinusoidal intensity pattern in the object plane is imaged in the focal plane
 - Modulation transfer function (MTF) is the amplitude part,
$$\text{MTF} = |\text{OTF}|$$
 - Phase transfer function (PTF) is the phase part,
$$\text{PTF} = \arg(\text{OTF})$$
- Each point in the object plane is blurred in the image plane by the point-spread-function (PSF), which is the Fourier transform of the OTF

$$i(x) = o(x) \otimes p(x)$$

$$I(f) = O(f)P(f)$$

Example: MTFs of varying amounts of aberration

