Mathematical Modelling in Adaptive Optics

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Outline of Talk

- Why do mathematical modelling?
- Illustrative Example: MEMS DM Model Development
  - Motivation
  - Conceptual Model
  - Mathematical Model
  - Numerical Solution to Model Equations
- Simulation Results
- What's Missing?

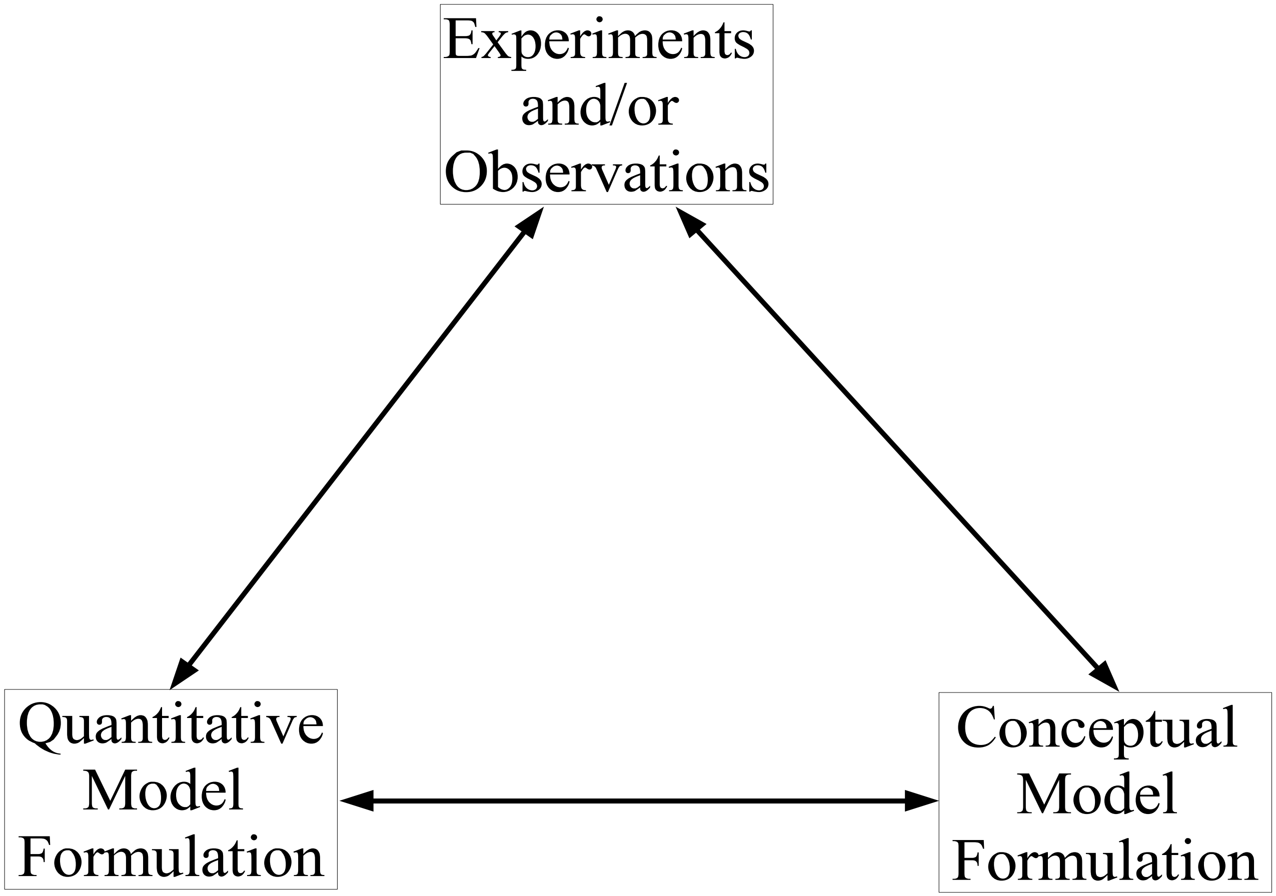
Why Do Mathematical Modelling?

"A couple of months in the laboratory can frequently save a couple of hours in the library"—a quote from Aron Ahmadia, who got it from David Keyes, who got it from Westheimer.

A good mathematical model can

- can provide insight into a physical process
- can provide quantitative information about the process
- can enable one to perform virtual experiments and test hypotheses
- can, if used properly, save lots of time and resources

Block Diagram of Model Development Process



Note that this process is nonlinear and iterative.

Motivation for MEMS DM Model Development

Killer Application: Multi-Object Adaptive Optics (MOAO)

Small movable mirrors, placed at locations of interest in the science field will be used to correct residual phase aberrations. Idea is that it may be easier to obtain excellent correction over a few small regions rather than uniformly good correction over a large field of view.

MOAO will require exquisitely accurate, open-loop control of small deformable mirrors.

Requirements for MOAO DMs

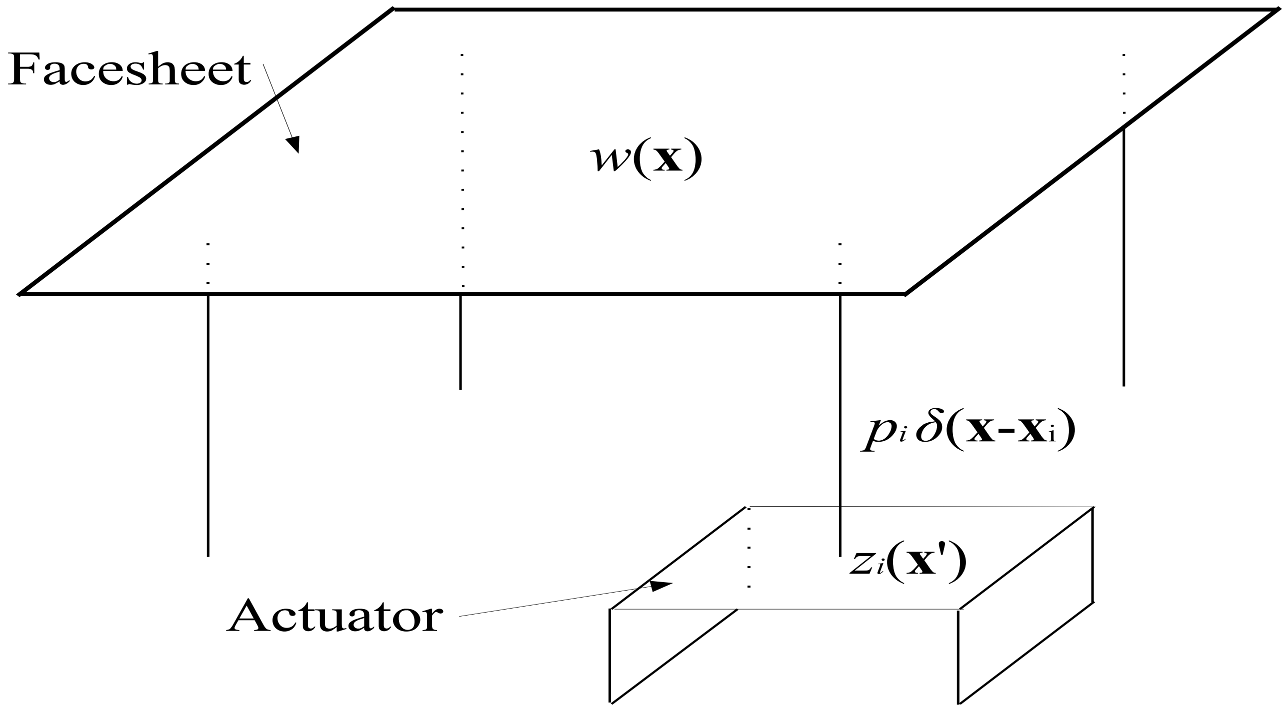
Extremely accurate (nanometer) open-loop DM control will require

- No hysteresis (don't need time-history of mirror position).
- Well-characterized, repeatable response to voltage inputs.
- Insensitivity to temperature changes.
- Continuous face sheet (almost no light scattering off DM surface—a requirement for ExAO).
- Fast (microsecond) damping time (a plus for ExAO control).

Electrostatically Actuated Plate-Plate DM Design

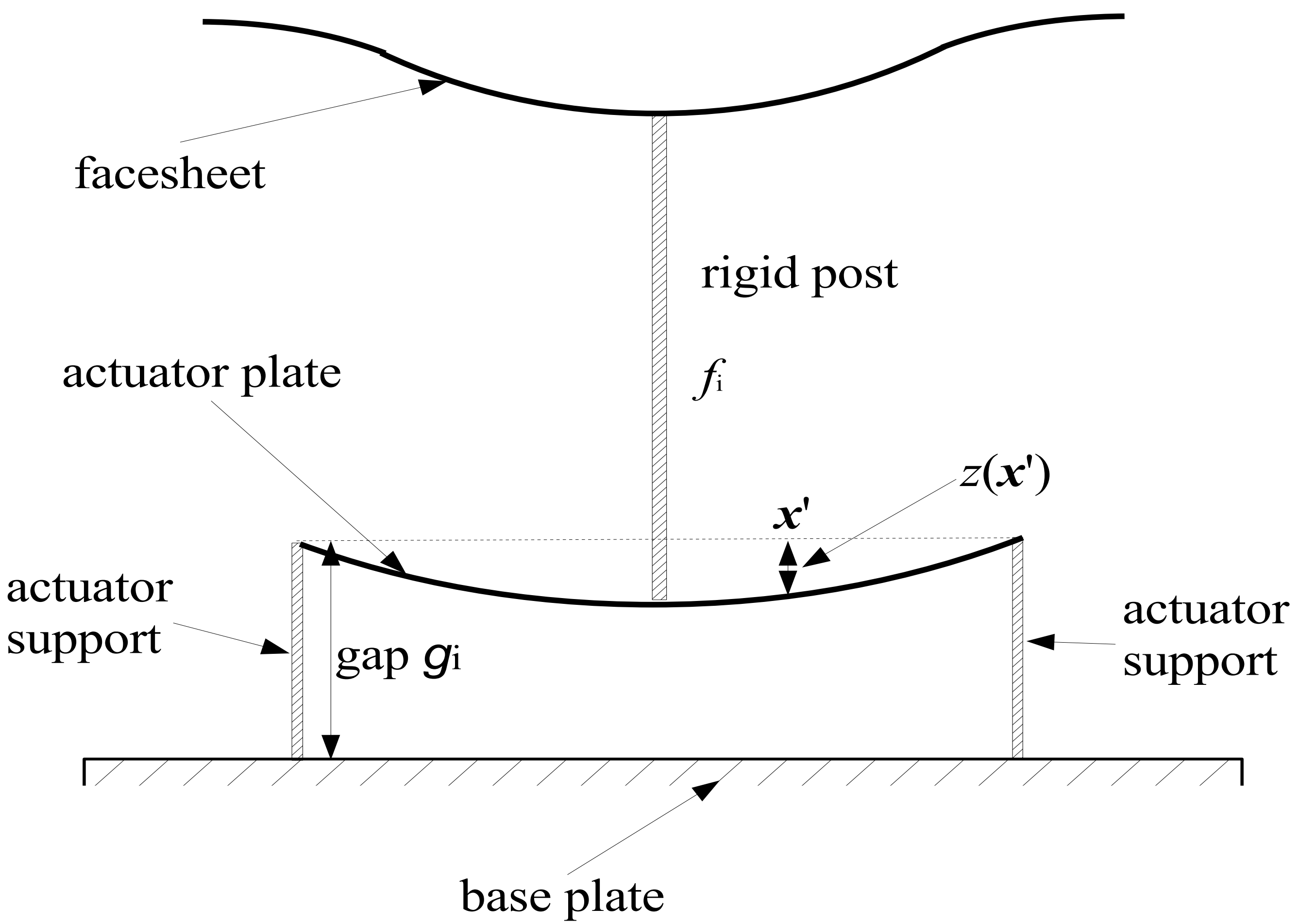
- Appears to have potential to meet all the MOAO requirements.
- Relatively easy to formulate an accurate mathematical model.

Schematic Diagram of Boston Micromachines' MEMS DM



- Facesheet is connected to actuator plates via rigid posts (vertical lines).
- Displacement (vertical position) of each actuator is controlled by adjusting voltage drop between actuator plate and base plate.
- Displaced actuator moves rigid post, which pulls on the facesheet.

Schematic Diagram for Actuator



Model for (Steady State) Facesheet Displacement

$$D_{\text{fs}} \nabla^4 w = -\sum_{i=1}^{n_a} p_i \delta(\mathbf{x} - \mathbf{x}_i)$$

- $w(\mathbf{x})$ denotes vertical displacement of facesheet at horizontal location $\mathbf{x} = (x, y)$.
- D_{fs} denotes flexural rigidity of the facesheet.
- p_i denotes load due to i^{th} actuator post; $\mathbf{x}_i$ denotes post location.
- $\delta(\cdot)$ denotes the Dirac Delta (idealized point loads).
- n_a denotes number of actuators.
- Biharmonic partial differential operator

$$\nabla^4 w \stackrel{\text{def}}{=} \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}.$$

Key idea in derivation: When the DM is at equilibrium, forces due to the posts pulling on the facesheet are balanced by restoring force due to flexure of the facesheet.

Models like this were developed by Euler and Bernoulli in the 1700's.

Mathematical Model for Actuator Displacement

$$\underbrace{D_{\mathrm{a}} \nabla^4 z_i}_{\text{plate restoring load}} = \underbrace{p_i \delta(\mathbf{x}' - \mathbf{0})}_{\text{load from post}} + \underbrace{\frac{\epsilon_0 \epsilon_r V_i^2}{(g - z_i(\mathbf{x}'))^2}}_{\text{electrostatic load}}$$

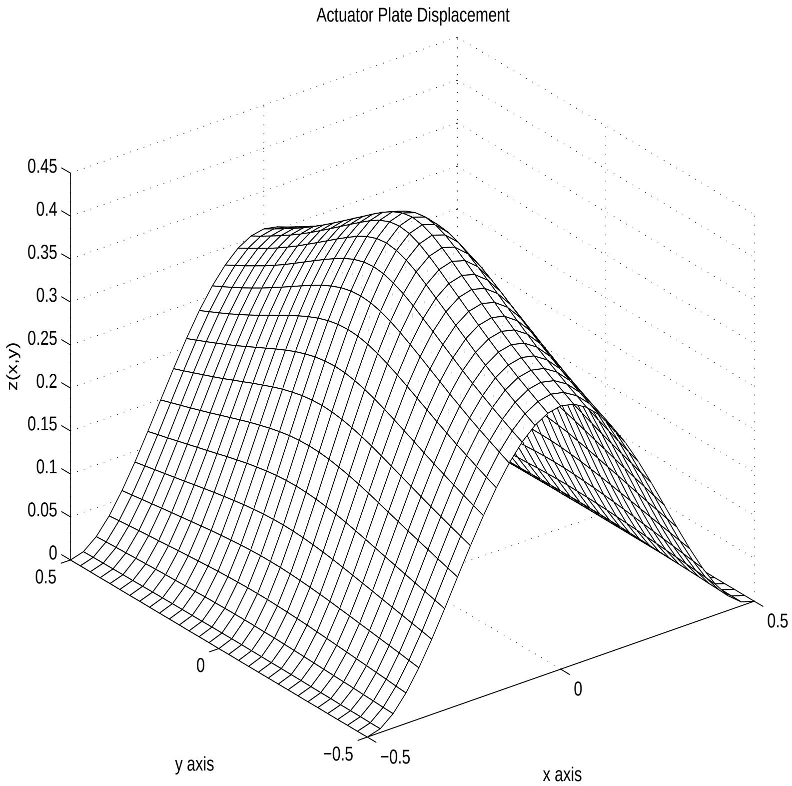
where

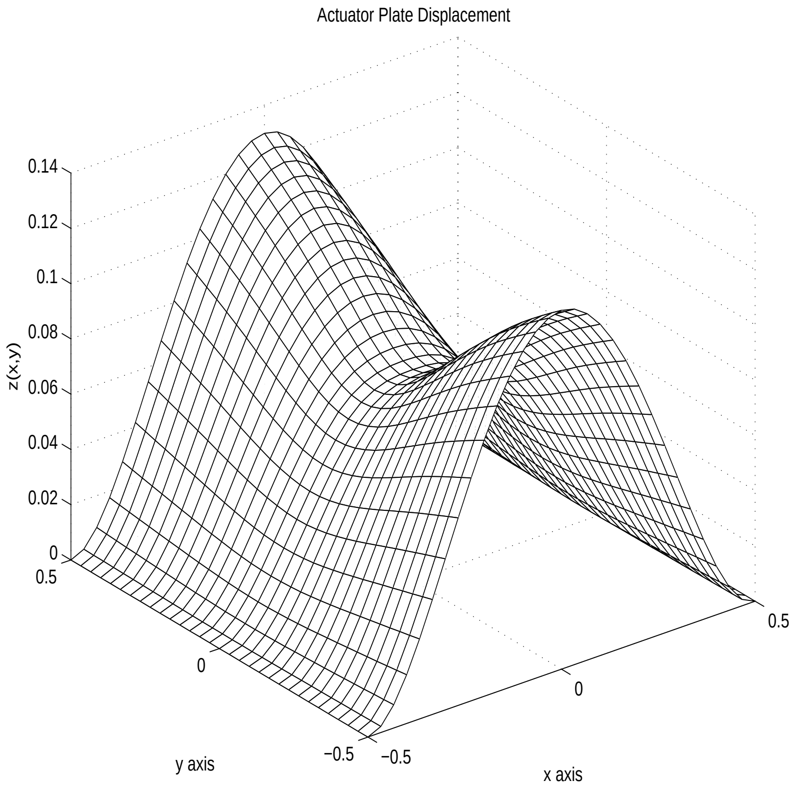
- $z_i(\mathbf{x}')$ is i^{th} actuator plate displacement at local coordinate location $\mathbf{x}'$.
- p_i is (again) the load due to the i^{th} post, which is connected to the facesheet.
- D_{a} is actuator plate flexural rigidity.
- ϵ_0 is the permittivity of free space.
- ϵ_r is dielectric constant of the gap media.
- g is the gap size of the actuator.
- V_i is the applied voltage.

The $(g - z_i)^{-2}$ nonlinearity comes from the $1/R^2$ power law for electrostatic force.

This is a modification of a model due to Bifano et al. Q. Yang and C.V. added the $p_i \delta(\mathbf{x}' - \mathbf{0})$ term to account for coupling to the facesheet from the rigid post.

Actuator Model Behavior





On the left, the load due to the post is positive; on the right, the post load is negative. Applied voltage (and electrostatic loading) is the same in both cases. Note that $+z$ is toward the baseplate.

Facesheet-to-Actuator Coupling Constraint

The displacement at the center of the i^{th} actuator plate (local location $\mathbf{x}' = \mathbf{0}$), where the rigid post attaches, matches the displacement of the facesheet at the i^{th} post location (facesheet location $\mathbf{x} = \mathbf{x}_i$).

$$w(\mathbf{x}_i) = z_i(\mathbf{0}), \quad i = 1, \ldots, n_a.$$

Conversion to Nondimensional Form

Why convert to nondimensional form?

- Resulting equations are independent of measurement system.
- Number of model parameters is reduced.
- Can derive scaling laws.

Nondimensional Form for the MEMS Model

Change of variables

- $\tilde{z}_i = z_i/g, \quad \tilde{\mathbf{x}}' = \mathbf{x}'/\ell$ (ℓ is actuator length)
- $\tilde{w} = w/g, \quad \tilde{x} = x/L$ (L is facesheet length)

applied to above model equations yields

$$\begin{aligned} \nabla^4 \tilde{z}_i &= \rho_i \delta(\tilde{\mathbf{x}}' - \mathbf{0}) + \frac{\nu_i}{(1 - \tilde{z}_i(\tilde{\mathbf{x}}'))^2}, \\ \nabla^4 \tilde{w} &= -\sigma \sum_{i=1}^{n_a} \rho_i \delta(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_i), \\ \tilde{w}(\tilde{\mathbf{x}}_i) &= \tilde{z}_i(\mathbf{0}), \quad i = 1, \ldots, n_a, \end{aligned}$$

where

$$\rho_i = \frac{\ell^4 p_i}{g D_a}, \qquad \nu_i = \frac{\epsilon_0 \epsilon_r \ell^4 V_i^2}{g^3 D_a}, \qquad \sigma = \left(\frac{L}{\ell}\right)^4 \frac{D_{\mathrm{a}}}{D_{\mathrm{fs}}}$$

- This reduces 7 parameter model to a 3 parameter model.
- σ is a scaling parameter which relates facesheet to actuator plate.

Model Reduction

We've derived a system of 1 linear partial differential equation (PDE) coupled to n_a nonlinear PDEs (1 PDE for each actuator) and n_a algebraic constraint equations. Solving this is a computational nightmare!

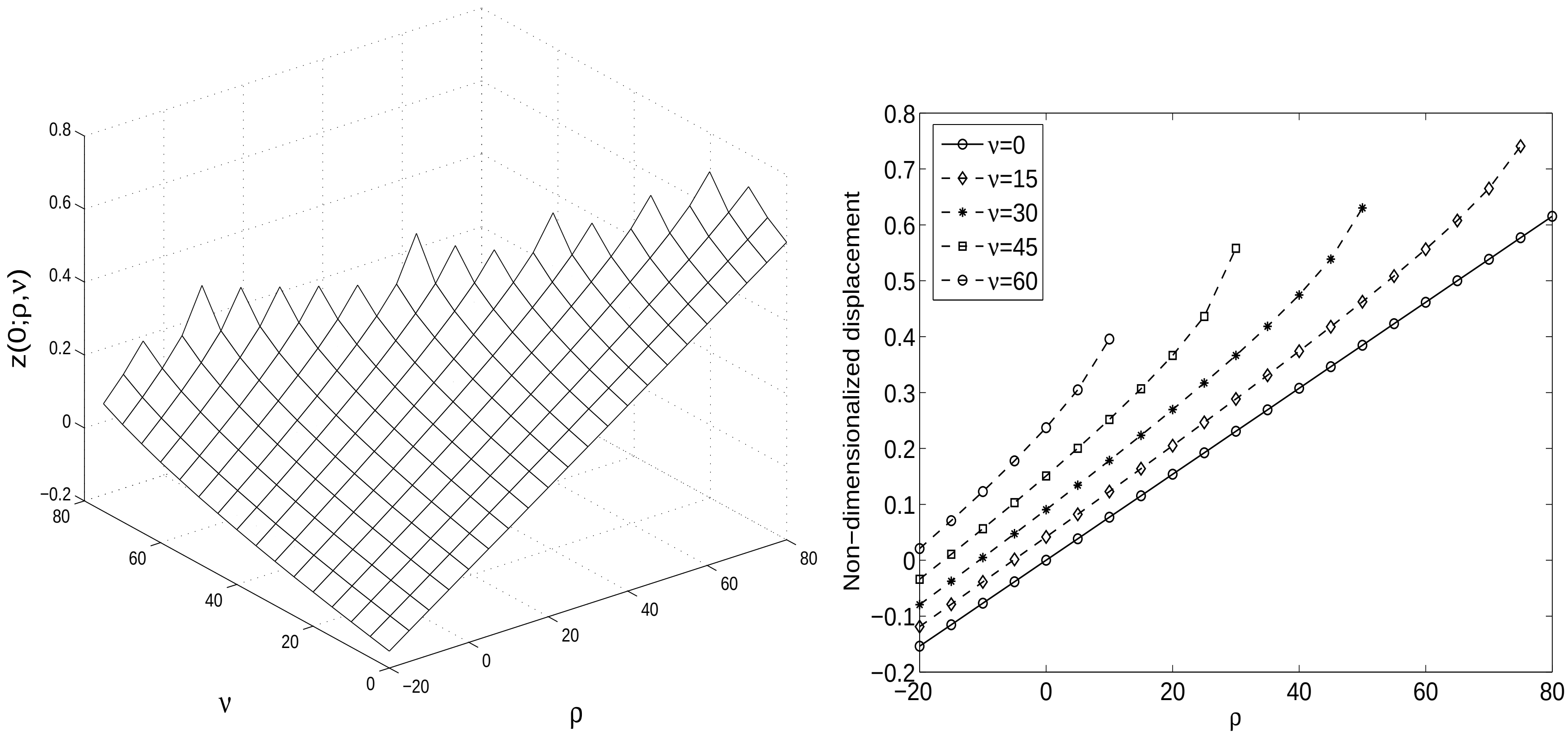
Key Idea for Model Reduction: We really don't need actuator plate displacement $z_i(\mathbf{x}')$ for all locations $\mathbf{x}'$; we only need displacement at the center (post location), $z_i(\mathbf{0})$. From the constraint, we get

$$\begin{aligned} w(\mathbf{x}_i) &= z_i(\mathbf{0}; \rho_i, \nu_i), \qquad i = 1, \ldots, n_a \\ &\stackrel{\text{def}}{=} f(\rho_i, \nu_i) \end{aligned}$$

Steps in construction of "actuator response function" $f(\rho, \nu)$.

1. For a range of values of ρ and ν, solve the actuator PDE to get $z(\mathbf{x}'; \rho, \nu)$.
2. Evaluate at the actuator center to get $f(\rho, \nu) = z(\mathbf{0}; \rho, \nu)$

Plots of the Actuator Response Function



- Recall that ν is proportional to square of applied voltage.
- ρ is proportional to load due to post.
- Note that f depends nonlinearly on ρ and ν.

Reduced MEMS DM Model

This consists of a single linear PDE

$$\nabla^4 w + \sigma \sum_{i=1}^{n_a} \rho_i \delta(\mathbf{x} - \mathbf{x}_i) = 0$$

coupled with n_a nonlinear algebraic constraints

$$w(\mathbf{x}_i) - f(\rho_i, \nu_i) = 0, \quad i = 1, \ldots, n_a.$$

This is computational tractable!

Numerical Solution to Reduced MEMS DM Model

Galerkin-Finite Element Solution: Assume expansion for facesheet displacement

$$w(\mathbf{x}) = \sum_{k=1}^{N} w_k b_k(\mathbf{x})$$

Pick basis functions $b_k(\mathbf{x})$ to be bicubic splines (bivariate piecewise polynomials).

Substitute expansion into linear PDE, multiply by $b_j(\mathbf{x})$, and integrate by parts to get

$$\sum_{k=1}^{N} \left(\int \nabla^2 b_j \nabla^2 b_k \, d\mathbf{x} \right) w_k + \sigma \sum_{i=1}^{n_a} \rho_i b_j(\mathbf{x}_i) = 0, \quad j = 1, \ldots, N.$$

Substitute expansion into the nonlinear constraint equations to get

$$\sum_{k=1}^{N} w_k b_k(\mathbf{x}_i) = f(\rho_i, \nu_i), \quad i = 1, \ldots, n_a.$$

Numerical Solution: Nonlinear System Formulation

Can express (discrete) finite element system as

$$\mathbf{G}(\mathbf{w},\boldsymbol{\rho}) = \begin{bmatrix} \mathbf{G}_1(\mathbf{w},\boldsymbol{\rho}) \\ \mathbf{G}_2(\mathbf{w},\boldsymbol{\rho}) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} A\mathbf{w} + \sigma B\boldsymbol{\rho} \\ B^T\mathbf{w} - \mathbf{f}(\boldsymbol{\rho},\boldsymbol{\nu}) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

where

$$\begin{aligned} [A]_{jk} &= \int \nabla^2 b_j \nabla^2 b_k \, d\mathbf{x}, \qquad j,k = 1,\ldots,N, \\ [B]_{ji} &= b_j(\mathbf{x}_i), \qquad j = 1,\ldots,N;\ i = 1,\ldots,n_a, \\ [\mathbf{f}(\boldsymbol{\rho},\boldsymbol{\nu})]_i &= f(\rho_i,\nu_i), \qquad i = 1,\ldots,n_a. \end{aligned}$$

Nonlinear System Solution Via Newton Iteration

Key Idea is Successive linearization.

At each iteration, solve linearized system

$$\begin{bmatrix} A & \sigma B \\ B^T & -D \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} -\mathbf{G}_1 \\ -\mathbf{G}_2 \end{bmatrix},$$

where

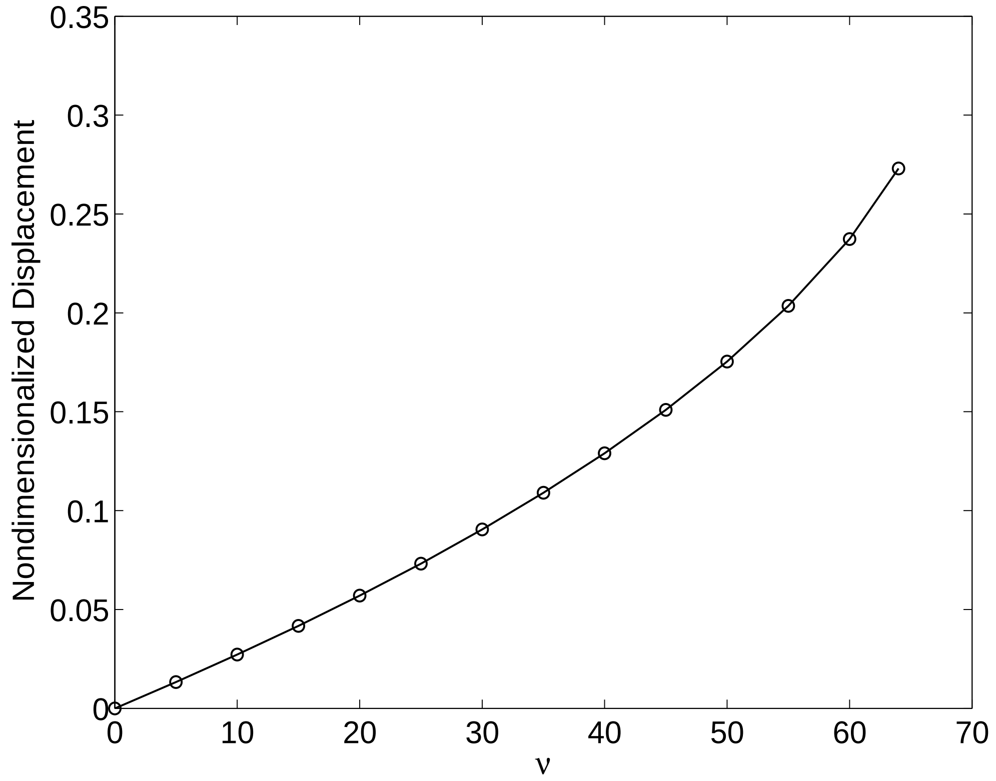
$$D = \mathrm{diag}\left(\frac{\partial f}{\partial \rho}(\rho_i, \eta_i) \right).$$

Then update solution,

$$\begin{aligned} \mathbf{w} &\leftarrow \mathbf{w} + \Delta \mathbf{w} \\ \boldsymbol{\rho} &\leftarrow \boldsymbol{\rho} + \Delta \boldsymbol{\rho} \end{aligned}$$

Simulation Results

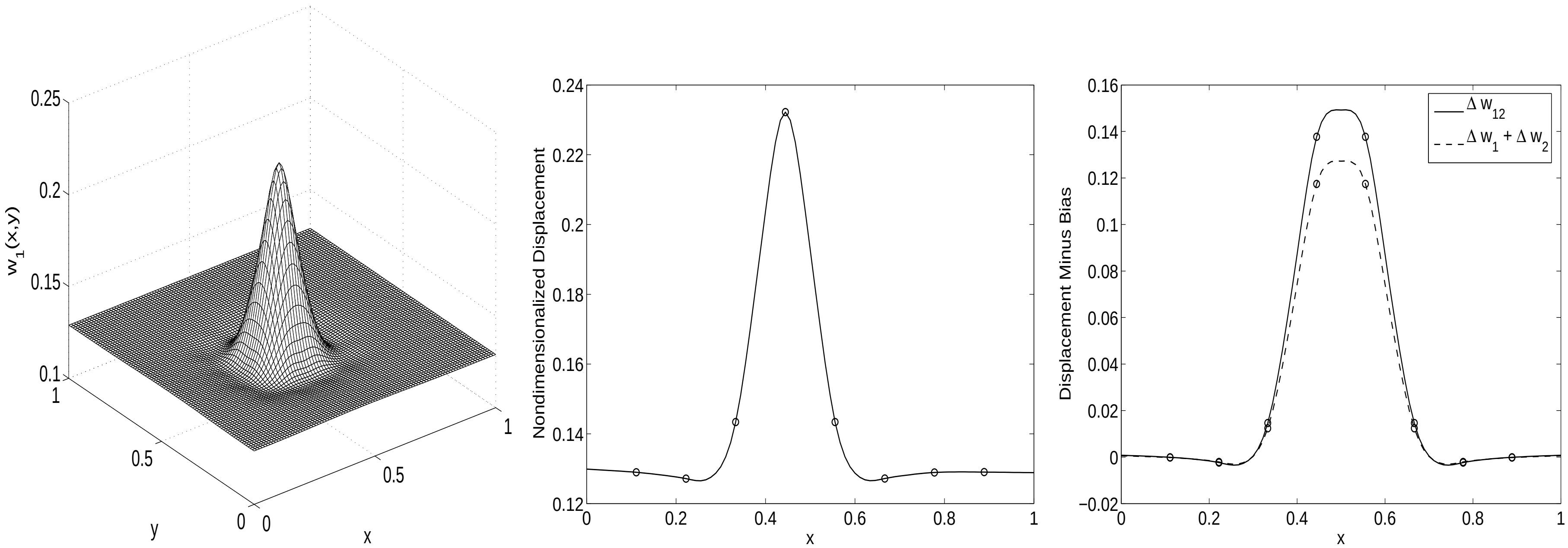
When the same voltage is applied to all actuators, the facesheet stays flat but is displaced (this corresponds to piston).



Recall that ν is proportional to squared voltage. At higher voltages, displacement becomes nonlinear with respect to ν. “Snap-down” occurs when voltage is too high.

Simulation Results, Continued

Facesheet displacement is not additive with respect to voltage applied to adjacent actuators.



What's Next?

- Model needs to calibrated against physical parameters and laboratory data.
- It is likely that the actuator response function varies with actuator index, i.e., need to replace $f(\rho_i, \nu_i)$ by $f_i(\rho_i, \nu_i)$. Open question: What sorts of experiments and/or measurements are needed to determine f_i?
- Given that we can calibrate the model, we still need to control the DM.

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