An overview of Fourier Optics

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A review of Fourier optics - outline

- Why it’s important – history, fundamental ideas
- Fourier transforms and properties
- The Fresnel and Fraunhofer approximations
- Treating a lens as a phase screen
- Fourier transforming properties of a lens
- Coherent imaging systems
- Incoherent imaging systems
Levels of models in optics

Geometric optics – rays, reflection, refraction

Fourier optics – diffraction, scalar waves, coherence

Electromagnetic optics – vector waves, polarization

Quantum optics – photons, interaction with matter
Key ideas of Fourier optics

Diffraction: “any deviation of light from rectilinear paths which cannot be interpreted as reflection or refraction” – Sommerfeld

1678 – Huygens introduced the idea that each point in a wavefront can be viewed as the source of a new spherical wave

1818 – Fresnel improves the idea by adding Young’s principle of interference

1882 – Kirchoff produces the mathematical basis

1894 – Sommerfeld improves the mathematical approach
Definition of Fourier Transform

- Fourier transform (or Fourier spectrum, frequency spectrum) of \( g(x,y) \) is defined as \( G \)

\[
\mathcal{F}\{g\} = \iint g(x, y) \exp[-2\pi i(f_x x + f_y y)] \, dx \, dy
\]

so \( \mathcal{F}\{g\} \) is a function of frequencies \( f_x \) and \( f_y \)

- The inverse formula is

\[
\mathcal{F}^{-1}\{G\} = \iint G(f_x, f_y) \exp[2\pi i(f_x x + f_y y)] \, df_x \, df_y
\]
• Range of applicability
  – g must be absolutely integrable over the infinite xy plane
  – g must have only a finite number of discontinuities and a finite number of maxima and minima in any finite rectangle
  – g must have no infinite discontinuities

  – These rules can often be stretched by taking limits in a suitable fashion as for δ functions

• One can view the Fourier transform as a decomposition, as in the inverse transform

\[ g(t) = \int_{-\infty}^{\infty} G(f) \exp(i2\pi ft) df \]
### Simple Functions and their Fourier transforms

<table>
<thead>
<tr>
<th>Function</th>
<th>Transform</th>
</tr>
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<tbody>
<tr>
<td>$\exp[-\pi(x^2 + y^2)]$</td>
<td>$\exp[-\pi(f_x^2 + f_y^2)]$</td>
</tr>
<tr>
<td>$\text{rect}(x)\text{rect}(y)$</td>
<td>$\sin c(f_x)\sin c(f_y)$</td>
</tr>
<tr>
<td>$\Lambda(x)\Lambda(y)$</td>
<td>$\sin^2(f_x)\sin^2(f_y)$</td>
</tr>
<tr>
<td>$\delta(x, y)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\exp[i\pi(x + y)]$</td>
<td>$\delta(f_x - 1/2, f_y - 1/2)$</td>
</tr>
<tr>
<td>$\text{sgn}(x)\text{sgn}(y)$</td>
<td>$\frac{1}{i\pi f_x} \frac{1}{i\pi f_y}$</td>
</tr>
<tr>
<td>$\text{comb}(x)\text{comb}(y)$</td>
<td>$\text{comb}(f_x)\text{comb}(f_y)$</td>
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Width in space (or time) and frequency are related through an uncertainty $\Delta x \Delta f = \text{constant}$.
Fourier transform properties

- **Linearity:** \( \mathcal{F}(\alpha g + \beta h) = \alpha \mathcal{F}(g) + \beta \mathcal{F}(h) \)
- **Scaling:**
  If \( \mathcal{F}\{g(x, y)\} = G(f_x, f_y) \), then
  \[ \mathcal{F}\{g(ax, by)\} = \frac{1}{|ab|} G\left(\frac{f_x}{a}, \frac{f_y}{b}\right) \]

- **Shift theorem:**
  \( \mathcal{F}\{g(x - a, y - b)\} = G(f_x, f_y) \exp[-i2\pi(f_x a + f_y b)] \)

- **Parseval’s theorem:**
  \[ \iint_{-\infty}^{\infty} |g(x, y)|^2 \, dx \, dy = \iint_{-\infty}^{\infty} |G(f_x, f_y)|^2 \, df_x \, df_y \]
Fournier transform properties-2

Convolution theorem:
If $\mathcal{F}\{g(x, y)\} = G(f_x, f_y)$
and $\mathcal{F}\{h(x, y)\} = H(f_x, f_y)$
then $\mathcal{F}\{\int_{\infty}^{-\infty} g(\xi, \eta)h(x - \xi, y - \eta)d\xi d\eta\} = G(f_x, f_y)H(f_x, f_y)$

Autocorrelation:
If $\mathcal{F}\{g(x, y)\} = G(f_x, f_y)$
and $\mathcal{F}\{h(x, y)\} = H(f_x, f_y)$
then $\mathcal{F}\{\int_{\infty}^{-\infty} g(\xi, \eta)g^*(\xi - x, \eta - y)d\xi d\eta\} = |G(f_x, f_y)|^2$
similarly, $\mathcal{F}\{|g(x, y)|^2\} = \int_{\infty}^{-\infty} G(\xi, \eta)G^*(\xi - f_x, \eta - f_y)d\xi d\eta$
Two dimensional linear systems

Input

\( g_1(x, y) \leftrightarrow G_1(f_x, f_y) \)

Linear shift-invariant system

\( h(x, y) \): impulse response or psf
\( H(f_x, f_y) \): transfer function

Output

\( g_2(x, y) \leftrightarrow G_2(f_x, f_y) \)

**Linearity:**
If \( g_1' \rightarrow g_2' \) and \( g_1'' \rightarrow g_2'' \)
then \( Ag_1' + Bg_1'' \rightarrow Ag_2' + Bg_2'' \)

**Shift-invariance:** Response to a point source \( h(x, y) \) does not depend on location of point sources (not generally true for optical systems)

**Input-output relations:**
\[
\begin{align*}
g_2(x, y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g_1(\xi, \eta) h(x - \xi, y - \eta) \, d\xi \, d\eta = g_1 * h \\
G_2(f_x, f_y) &= G_1(f_x, f_y) H(f_x, f_y)\end{align*}
\]
The initial step in the evolution of a theory that would explain such effects was made by the first proponent of the wave theory of light, Christian Huygens, in the year 1678. Huygens expressed an intuitive conviction that if each point on the wavefront of a light disturbance were considered to be a new source of a "secondary" spherical disturbance, then the wavefront at any later instant could be found by constructing the "envelope" of the secondary wavelets, as illustrated in Fig. 3-2.
Notation for scalar waves

Assume a monochromatic wave with electric field given by

\[ u(P,t) = U(P)\cos[2\pi vt + \phi(P)] \]

Where \( U(P) \) and \( \phi(P) \) are amplitude and phase of the wave at position \( P \) and \( \nu \) is the optical frequency.

We can write the field in complex notation as

\[ u(P,t) = \text{Re}\left[ U(P)e^{-i2\pi vt} \right] \]

where \( U(P) \) is the phasor

\[ U(P) = U(P)e^{-i\phi(P)} \]

We will generally work with the phasor \( U(P) \) – the harmonic dependence on time is understood.
Point source illumination of a plane screen

\[ P_1, r_{01}, z \]
• With the wave equation and Greens theorem it can be shown that

\[ U(P_0) = \frac{1}{i\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(ikr_{01})}{r_{01}} \cos(n, r_{01}) \, ds \]

– Where \( U(P) \) is the field at \( P \)

• Or, rewriting this in the form of a superposition integral,

\[ U(P_0) = \iint_{\Sigma} h(P_0, P_1) U(P_1) \, ds \]
Fresnel Diffraction

• Approximation leading to simplification
  – From above formulas, assume limits of integration are $\infty$ where $U$ is identically zero outside of the aperture.
  – Distance to observation plane is very large compared to aperture
  – Observing region is close to on axis, so $\cos(n,r) \approx 1$
  – This also implies that $r$ does not change significantly

• Simplified formula
  
  $h(x_0, y_0; x_1, y_1) \approx \frac{1}{i\lambda z} \exp(ikr_{01})$
  – Note $r$ cannot be replaced by a constant in the exponent.
  – For the exponent, expand $r$
Fresnel Diffraction-2

\[ r_{01} = z \sqrt{1 + \left( \frac{x_0 - x_1}{z} \right)^2 + \left( \frac{y_0 - y_1}{z} \right)^2} \]

\[ \approx z \left[ 1 + \frac{1}{2} \left( \frac{x_0 - x_1}{z} \right)^2 + \frac{1}{2} \left( \frac{y_0 - y_1}{z} \right)^2 \right] \]

This is called the Fresnel approximation, and allows \( h \) to be written as

\[ h(x_0, y_0; x_1, y_1) \equiv \frac{\exp(ikz)}{i\lambda z} \exp \left\{ \frac{ik}{2z} \left[ (x_0 - x_1)^2 + (y_0 - y_1)^2 \right] \right\} \]

\[ U(x_0, y_0) = \frac{\exp(ikz)}{i\lambda z} \int_\infty \int U(x_1, y_1) \exp \left\{ \frac{ik}{2z} \left[ (x_0 - x_1)^2 + (y_0 - y_1)^2 \right] \right\} dx_1 dy_1 \]
Which has the form of a convolution. Or it can be rewritten to look like a Fourier transform

\[ U(x_0, y_0) = \frac{\exp(ikz)}{i\lambda z} \exp \left[ \frac{ik}{2z} (x_0^2 + y_0^2) \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_1, y_1) \exp \left[ \frac{ik}{2z} (x_1^2 + y_1^2) \right] \exp \left[ -\frac{2\pi i}{\lambda z} (x_0 x_1 + y_0 y_1) \right] dx_1 dy_1 \]

- Where the Fourier transform is of the aperture function \( U \) times a quadratic phase factor

- Fresnel diffraction is sometimes called “near field”, and Fraunhofer diffraction “far field”
• If more stringent restrictions are applied

\[ z >> \frac{k(x_1^2 + y_1^2)_{\text{max}}}{2} \]

— Then the quadratic phase factor is approximately unity over the aperture, and we can write

\[
U(x_0, y_0) = \frac{\exp(ikz)}{i\lambda z} \exp\left[ \frac{ik}{2z} (x_0^2 + y_0^2) \right] \iint U(x_1, y_1) \exp\left[ \frac{-2\pi i}{\lambda z} (x_0 x_1 + y_0 y_1) \right] dx_1 dy_1
\]

— So now the field looks like the Fourier transform of the aperture evaluated at

\[
f_x = \frac{x_0}{\lambda z}, \quad f_y = \frac{y_0}{\lambda z}
\]
• We want the intensity pattern caused by a rectangular aperture $L_x \times L_y$. So the transmission function is $t(x_1, y_1) = \text{rect}\left(\frac{x_1}{L_x}\right)\text{rect}\left(\frac{y_1}{L_y}\right)$.

So,

$$U(x_0, y_0) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{ik}{2z} (x_0^2 + y_0^2)\right] \mathcal{F}\{U(x_1, y_1)\}_\text{aperture}$$

and $\mathcal{F}\{U(x_1, y_1)\}_\text{aperture} = L_x \cdot L_y \sin c(L_x f_x) \sin c(L_y f_y)$,

yielding,

$$I(x_0, y_0) = \frac{L_x^2 L_y^2}{\lambda^2 z^2} \sin c^2\left(\frac{L_x x_0}{\lambda z}\right) \sin c^2\left(\frac{L_y y_0}{\lambda z}\right)$$
Examples of far-field diffraction from a square aperture
We want the intensity pattern caused by a circular aperture diameter \( D \). So the transmission function is

\[
\textit{t}(r_1) = \text{circ} \left( \frac{2r_1}{D} \right)
\]

So,

\[
U(r_0) = \frac{\exp(ikz)}{i\lambda z} \exp \left( \frac{ikr_0^2}{2z} \right) \mathcal{B}\{U(r_1)\} \big|_{\rho = r_0 / \lambda z}
\]

\[
U(r_1) = \textit{t}(r_1) \text{ for unit amplitude illumination}
\]

\[
\mathcal{B}\{\text{circ} \left( \frac{2r_1}{D} \right) \} = \frac{2D^2 J_1(\pi D \rho)}{D \rho}
\]

yielding

\[
I(r_0) = \left( \frac{kD^2}{8z} \right)^2 \left[ \frac{2J_1(kD r_0 / 2z)}{kD r_0 / 2z} \right]^2
\]
Examples of diffraction from a circular aperture
Near-field propagation at increasing distance from a 20 cm circular aperture
Atmospheric simulation model

1) Start by initializing a set of phase screens with Komolgorov turbulence. Each screen represents a length \( dL_z \) (typically 100 m) of atmosphere, compressed into a thin slab of "equivalent phase." Light propagates between screens as if in vacuum.

2) Define a laser source at one end, and (instantaneously) propagate light from one screen to the next.

3) When light hits a screen, it's phase, \( F_L(x,y) \), is modified by the effective phase present on the screen, \( F_{SCR}(x,y) \).

4) Aperture the last screen, by receiver area, to obtain received signal.

This type of simulation has been in use for over 25 years* for laser guide star applications. We have simply modified it for horizontal path.

$C_n^2 = 0.8 \times 10^{-13} \text{ m}^{-2/3}$, $D = 0.2 \text{ m}$, $\lambda = 1.0 \text{ \mu m}$
\[ C_n^2 = 0.8 \times 10^{-13} \, \text{m}^{-2/3}, \ D = 0.2 \, \text{m}, \ l = 1.0 \, \text{mm} \]
\[ C_n^2 = 0.8 \times 10^{-13} \text{ m}^{-2/3}, \quad D = 0.2 \text{ m}, \quad \lambda = 1.0 \mu\text{m} \]
A thin lens can be modeled as a phase screen

The field immediately behind the lens is

\[ U^*(x, y) = t_1(x, y)U(x, y) \]

where

\[ t_1(x, y) = e^{i\Delta_0} e^{ik(n-1)\Delta(x, y)} \]

and \( U(x, y) \) is the incident field. \( \Delta(x, y) \) is the lens thickness at point \( (x, y) \) given by

\[ \Delta(x, y) = \Delta_0 - \frac{x^2 + y^2}{2} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \]

Using the definition for focal length

\[ \frac{1}{f} = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \]

the lens transformation is

\[ t_1(x, y) = e^{ik\Delta_0} e^{-ik/2f(x'^2+y'^2)} \]

Plane wave \( \rightarrow \) spherical wave focused at distance \( f \)
Fourier transform property of a lens

Field incident on lens: \( U_i(x, y) = t(x, y) \)

Field behind lens: \( U'_i(x, y) = t(x, y) e^{-i \frac{k}{2f} (x^2 + y^2)} \)

Propagate over distance \( f \) using Fresnel diffraction

\[
U_f(x_0, y_0) = \frac{e^{i \frac{k}{f} (x_0^2 + y_0^2)}}{i \lambda f} \iint_{ap} U'_i(x, y) e^{i \frac{k}{f} (x^2 + y^2)} e^{-i \frac{2\pi}{\lambda f} (xx + yy)} \, dx \, dy
\]

Note that the quadratic phases in the lens model and the Fresnel propagation cancel leaving

\[
U_f(x_0, y_0) = \frac{e^{i \frac{k}{f} (x_0^2 + y_0^2)}}{i \lambda f} \iint_{ap} t(x, y) e^{-i \frac{2\pi}{\lambda f} (xx + yy)} \, dx \, dy
\]

which is the Fourier transform of \( t(x, y) \) – multiplied by a quadratic phase.
Imaging a point source with a lens

1. Point source generates spherical wave $u_o$. Compute the field $u_i$ at the lens using Fresnel propagation.
2. Multiply $u_i$ by lens transfer function to get transmitted field $u'_i$.
3. Fresnel propagate by $z_2$ to get image plane field $u_i$.

Result is:

$$h(x_i, y_i; x_o, y_o) = \frac{1}{\lambda^2 z_1 z_2} e^{\frac{ik}{2} (x_i^2 + y_i^2)} e^{\frac{ik}{2} (x_o^2 + y_o^2)} \int \int \exp \left[ \frac{ik}{2} \left( \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f} \right) \left( x^2 + y^2 \right) \right] P(x, y) \left[ -ik \left( \frac{x_o}{z_1} + \frac{x_i}{z_2} \right) x + \left( \frac{y_o}{z_1} + \frac{y_i}{z_2} \right) y \right] dx dy$$

First quadratic phase terms are there because we are imaging between spherical surfaces. Generally this is not important for imaging between planes:
- phase curvature in $U_i$ plane does not affect intensity
- image intensity depends on object phase over a small region → for any image point the object phase curvature is ~ constant
Imaging a point source (continued)

To get rid of the first exponential in the integral we choose distances $z_1$, $z_2$ so

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}$$

giving us the lens law from geometric optics. Defining the system magnification as $M = \frac{z_2}{z_1}$, the point source image is

$$h(x_i, y_i; x_o, y_o) \approx \frac{1}{\lambda^2 z_1 z_2} \iint_{ap} P(x, y) \exp \left[ -\frac{i 2 \pi}{\lambda z_2} \left( (x_i + M x_o) x + (y_i + M y_o) y \right) \right] dx \, dy$$

This is the Fraunhofer diffraction pattern of the lens aperture centered at image coordinates (-M$x_o$,-M$y_o$)
Coherent transfer function

We can write the amplitude as a superposition

\[ U(x_i, y_i) = \iint_{0}^{\infty} h(x_i, y_i; x_0, y_0) U_g(x_0, y_0) \, dx_0 \, dy_0 \]

— Where \( h \) is the amplitude at image point \( i \) in response to a point-source object \( 0 \)

When the system is space invariant or isoplanatic,

\[ h(x_i, y_i; x_0, y_0) = h(x_i - x_0; y_i - y_0) \]

Then we can write the amplitude as a convolution

\[ U(x_i, y_i) = \iint_{0}^{\infty} h(x_i - x_0; y_i - y_0) U_g(x_0, y_0) \, dx_0 \, dy_0 \]
Coherent transfer function-2

Writing down the Fourier transforms
\[ G_g (f_X, f_Y) = \int_\infty \int \mathcal{U}_g (x_0, y_0) \exp[-i2\pi(f_X x_0 + f_Y y_0)] \, dx_0 \, dy_0 \]
\[ G_i (f_X, f_Y) = \int_\infty \int \mathcal{U}_i (x_i, y_i) \exp[-i2\pi(f_X x_i + f_Y y_i)] \, dx_i \, dy_i \]
\[ H(f_X, f_Y) = \int_\infty \int h(x_i, y_i) \exp[-i2\pi(f_X x_i + f_Y y_i)] \, dx_i \, dy_i \]

We apply the convolution theorem to get
\[ G_i (f_X, f_Y) = H(f_X, f_Y) G_g (f_X, f_Y) \]

H is called the coherent transfer function and is equal to the pupil function P
For incoherent systems we can obtain
\[ I_i(x_i, y_i) = \kappa \iiint_{-\infty}^{\infty} |h(x_i - x_0; y_i - y_0)|^2 I_g(x_0, y_0) \, dx_0 \, dy_0 \]

And then define the normalized Fourier transforms
\[ G_g(f_X, f_Y) = \frac{\iiint_{-\infty}^{\infty} I_g(x_0, y_0) \exp[-i2\pi(f_X x_0 + f_Y y_0)] \, dx_0 \, dy_0}{\iiint_{-\infty}^{\infty} I_g(x_0, y_0) \, dx_0 \, dy_0} \]
\[ G_i(f_X, f_Y) = \frac{\iiint_{-\infty}^{\infty} I_i(x_i, y_i) \exp[-i2\pi(f_X x_i + f_Y y_i)] \, dx_i \, dy_i}{\iiint_{-\infty}^{\infty} I_i(x_i, y_i) \, dx_i \, dy_i} \]
\[ H(f_X, f_Y) = \frac{\iiint_{-\infty}^{\infty} |h(x_i, y_i)|^2 \exp[-i2\pi(f_X x_i + f_Y y_i)] \, dx_i \, dy_i}{\iiint_{-\infty}^{\infty} |h(x_i, y_i)|^2 \, dx_i \, dy_i} \]
To obtain

\[
G_i(f_X, f_Y) = H(f_X, f_Y)G_g(f_X, f_Y)
\]

\(H\) is known as the optical transfer function of the system

Note that \(H\) is related to the coherent transfer function \(h\), and if we use \(H = \mathcal{F}\{h\}\), one can show
General properties of OTF

Basic equation of OTF: $H(f_x, f_y)$

$$H(f_x, f_y) = \frac{\iint H(\xi + \frac{f_x}{2}, \eta + \frac{f_y}{2})H^*(\xi - \frac{f_x}{2}, \eta - \frac{f_y}{2})d\xi d\eta}{\iint \|H(\xi, \eta)\|^2 d\xi d\eta}$$

Important Properties

- $H(0,0) = 1$
- $H(-f_x, -f_y) = H^*(f_x, f_y)$
- $|H(f_x, f_y)| < |H(0,0)|$
Optical transfer function for 0.5 m circular pupil with obscuration
Summary - What you should remember

- Huygens principle- decomposition in spherical wavelets
- Fresnel approximation → parabolic wavelets → near field
- Frammhofoa approximation → plane wavelets → far field
- A lens is a quadratic phase
- Pupil & image planes are related by a Fourier transform
- The coherent transfer function is the pupil function
- The incoherent transfer function (OTF) is the auto correlation of the pupil function
- The point-spread-function (psf) is the Fourier transform of the transfer function