Optimal modal Fourier transform wave-front Control

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We want fast and smart algorithms

§ Large systems and high frame rates require fast algorithms:
  § Matrix-based methods, which apply the control matrix more quickly
    § Local control, sparse matrix methods, conjugate gradient & multigrid
  § Fourier transform reconstruction (FTR), which filters the slopes

§ Our controller should use knowledge about the phase aberration and noise to its advantage:
  § Can use a priori model of atmosphere and noise
    § MVU, weighted least-squares
  § Can incorporate real-time information about the phase aberration and noise level as determined from telemetry of AO system
What is modal control?

§ For n actuators, create an orthonormal basis of n functions
  § these basis functions are normally chosen to concentrate signal power in few modes
§ Instead of controlling the phase value at each actuator, we control the amount of each mode present
  § this is expressed as the modal coefficient
§ The AO control problem simplifies to finding the optimal control law for each mode, independently of all the others

§ Explicit modal control with optimization is used in Altair and ESO’s NAOS
Optimal Fourier Control is our solution

§ Our modal set is the Fourier basis. This works even on an arbitrary aperture.
§ Reconstruction at each time step is with FTR.
§ Closed-loop modal coefficients are used to estimate optimal gains for control law for each mode. Gains are implemented as a filter.
§ Computationally feasible for 64x64 ExAO right now.
§ Extra benefits include
  § Modal coefficients are available for free, unlike matrix-based modal control, which requires extra computation.
  § There is a natural relationship between filter structure and PSF structure.
FTR works by filtering the slopes

WFS slopes

Fix boundary problem

FFT

Filter

FFT\(^{-1}\)

Phase estimate
The Fourier basis is the modal set

- FTR uses the DFT on N x N real signals. This leads to a real cosine and sine ONB with N^2 modes:
  - For \([k,l] \) either \([0,0]\), \([0,N/2]\), \([N/2,0]\) or \([N/2,N/2]\), there is only a cosine mode
    \[
    C_{k,l}[m,n] = \frac{1}{N} \cos \left( \frac{2\pi}{N} [km + ln] \right)
    \]
  - All other frequencies have a sine and cosine
    \[
    C_{k,l}[m,n] = \frac{\sqrt{2}}{N} \cos \left( \frac{2\pi}{N} [km + ln] \right)
    \]
    \[
    S_{k,l}[m,n] = \frac{\sqrt{2}}{N} \sin \left( \frac{2\pi}{N} [km + ln] \right)
    \]
Use the DFT to get modal coefficients

§ FTR uses the DFT

\[ X[k, l] = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x[m, n] \exp \left( -j2\pi \frac{(km + ln)}{N} \right) \]

§ Modal coefficients are obtainable directly from the DFT

\[ < x[m, n], C_{k,l}[m, n] > = \frac{1}{D_{k,l}} \text{Re} \{ X[k, l] \} \]

\[ < x[m, n], S_{k,l}[m, n] > = \frac{-1}{D_{k,l}} \text{Im} \{ X[k, l] \} \]
Modes correspond to PSF locations

§ Each Fourier mode lives at a specific spatial frequency pair \([k,l]\)

§ Because the PSF is approximately the PSD of the residual phase (to second order), each Fourier mode appears at a specific location in the PSF
Modes are eigenfunctions

§ Fourier modes are eigenfunctions of linear, shift-invariant (LSI) systems

§ The modes for the slopes (on a square aperture) are the same as the modes for the phase

§ A cosine of phase at frequency \([k,l]\) produces x- and y-slopes only at the cosine and sine of that frequency \([k,l]\)

§ Where \(M_x[k,l]\) describes the filter which measures the x-slopes from actuator commands

\[
\begin{align*}
\text{phase} & \quad \text{x-slope} \\
C_{k,l}[m,n] & \quad AC_{k,l}[m,n] + BS_{k,l}[m,n] \\
S_{k,l}[m,n] & \quad -BC_{k,l}[m,n] + AS_{k,l}[m,n]
\end{align*}
\]

\[
A = \text{Re}\{M_x[k,l]\}, \quad B = -\text{Im}\{M_x[k,l]\}
\]
Filter comes from modal responses

§ We simply pseudo-invert the measurement matrix

\[
\begin{bmatrix}
A & -B \\
B & A \\
C & -D \\
D & C
\end{bmatrix}
\]

x-slope cosine and sine  
y-slope cosine and sine  

§ And obtain the reconstruction matrix for the phase modes as:

\[
\frac{1}{A^2 + B^2 + C^2 + D^2} \begin{bmatrix}
A & B & C & D \\
-B & A & -D & C
\end{bmatrix}
\]

§ This produces an extremely sparse modal control matrix
滤波器反转测量过程

§ 滤波器由模态系数推导...

\[ \hat{P}[k, l] = \frac{(A + jB)X[k, l] + (C + jD)Y[k, l]}{|(A + jB)|^2 + |(C + jD)|^2} \]

§ ...与如果我们知道测量滤波器完全相同

\[ \hat{P}[k, l] = \frac{M_x^*[k, l]X[k, l] + M_y^*[k, l]Y[k, l]}{|M_x[k, l]|^2 + |M_y[k, l]|^2} \]
Many filtering options now available

§ Best point-based model filter is Modified-Hudgin

§ Assuming ideal continuous models for WFS and DM, can derived the Ideal filter

§ Ideal filter and Mod-Hud very similar

§ Given any LSI AO system or simulation, we can measure the coefficients that describe the modal responses and produce a Custom filter

§ captures influence function response of DM

§ Each filter exactly reconstructs given the assumed model, except for invisible modes of piston and sometimes waffle.
We can do all this with an aperture

§ Fourier basis in an arbitrary aperture is a tight Frame that allows analysis and synthesis like an ONB.

§ If we window the data, we can use a fast DFT to get modal coefficients.

§ New method of slope management called edge correction ensures high-quality coefficient estimation by making outside region of phase flat.

§ Result - we get the modal coefficients for free at each time step in the FTR process.
§ We follow Altair’s implementation and assume an approximate model of control system (exact in simulation case) for each of the independent modes.

§ We control a mode with feedback in the presence of noise.

Block diagram of control loop for a modal coefficient
Optimize the squared-residual error

Since the noise at any step is independent of past errors, if we minimize on the measurement $s$, we minimize on the residual error.

If we had perfect knowledge we would minimize

$$\mathcal{J} = \int \left| \frac{1}{1 + \exp(-j\omega)H(\omega)} \right|^2 [M(\omega) + N(\omega)] \, d\omega$$

But we don’t... so we have to estimate the open-loop PSD from the closed-loop measurements using

$$\hat{M}(\omega) + \hat{N}(\omega) = |1 + \exp(-j\omega)H_0(\omega)|^2 \hat{S}(\omega)$$
Gain estimation for FTR (1)

§ From closed-loop telemetry, we estimate the closed-loop measurement PSDs
§ Convert these to open-loop PSD estimates
§ Find the control law which minimizes the error for the sine and cosine modes together

\[
\arg\min H(z) \left\{ \int \left| \frac{1}{1 + \exp(-j\omega)H(\omega)} \right|^2 \left| 1 + \exp(-j\omega)H_0(\omega) \right|^2 \left[ \hat{S}_S(\omega) + \hat{S}_C(\omega) \right] d\omega \right\}
\]

§ Where our control law is simple: 
\[
H(z) = \frac{g}{1 - cz^{-1}}
\]
§ For a single variable (gain g) we can solve the optimization problem efficiently.

§ At each frequency \([k,l]\) we have a gain - we construct the filter of these gains using Hermitian symmetry. This filter in then incorporated into the reconstruction filter.
Gains are incorporated into filter

$WFS \text{ slopes}$

$\text{Fix boundary problem}$

$\text{FFT}$ $\rightarrow$ $\text{Filter}$ $\rightarrow$ $\text{FFT}^{-1}$

$\text{Phase estimate}$
Gains are incorporated into filter

WFS slopes

Fix boundary problem

FFT

Gain optimizer

Filter

FFT⁻¹

Phase estimate
Details of end-to-end ExAOC simulation

§ Features of ExAOC simulation include:

§ Fourier Optics for Spatially-filtered WFS onto CCD, quadcell config
§ Altair-based DM model using influence functions
§ Input phase aberration is a very long screen shifted at 10m/s
§ FTR reconstruction
§ Modal coefficients obtained in reconstruction stage
§ Gain optimization as describe above
§ Full diagnostics including long-exposure PSDs and PSFs from the residual wavefront and instantaneous residual error in different spatial frequency bands
§ Run either single long case to watch optimization or many short cases with a specific filter to analyze general case performance
**Significant reduction in residual error**

- Use of optimal gains improves performance
- Significant reduction in residual MSE at each timestep
- Less variation in MSE at each timestep

N=48, NGS Mag 8 example for 8 iterations of gain optimization
Comparison shows improvement

§ N=48 case with WFS SNR of 2.16
§ Strehl increased from 0.75 to 0.87 (+12%)
§ MSE in band reduced from 0.224 to 0.074 (3 times less)
Trade bandwidth and sensor errors

At high SNRs, optimal gains produce equivalent or more measurement error but less temporal error than before.

At low SNRs, optimal gains produce less measurement error but more temporal error than before.

Data for N=48, median over a set of 25 random phase screens
We can control modes independently

Given an optimal gain profile, we compare three filters

1. constant gain of 0.6 for all modes
2. optimized gains
3. constant gains with optimized for a smaller region of filter

PSD of case (3) is almost exactly the combination of parts of the other two responses in the right places
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- PSD of case (3) is almost exactly the combination of parts of the other two responses in the right places.
Optimal gains compensate for DM

§ Test: run Mod-Hud and Custom filter on same aberration and determine steady-state optimal gains

§ Ratio of these gains is almost exactly the inversion of the DM response, as determined empirically from AO system models

Inversion of DM influence function
Computational load is satisfiable today

§ FTR each timestep: \(15N^2 \lg N + 20N^2\)

§ Estimating periodograms for \(t\) steps of telemetry:

\[
N^2 (5 + 2.5 \lg t)
\]

§ Averaging the periodograms and finding the optimal gain (\(k\) is for evaluations in root-finding):

\[
N^2 (1 + k) + 4k
\]

§ Assuming \(k = 10\) (using fast method), a 64x64 system at 2.5k kHz has a maximum load of 1.43 GFLOPs/sec.
Plans for Optimal Fourier Control

§ Short term continuation of theory:
  § Explore complex gain filters and higher-order control laws
  § Contribute to ExAOC system design with performance predictions
  § Verify system measurement procedure (custom filters) at LAO ExAO testbed

§ Long term experimental verification of performance:
  § Implement at LAO testbed in ExAOC control system

§ Paper preprint (PDF) available