Generalized Wave Front Reconstruction Algorithm for slope data

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Outline

1. Mathematical Model
2. Domain Extension technique & Generalized WFR matrix equations
3. Solutions
4. Error analysis
5. Deviation error removal
6. Summary
7. References

Motivations:

Have you tied of wavefront reconstruction? Especially when the sampling pupil is large and irregular or even changing?

Doing a WF reconstruction is laborious and time-consuming!

Motivations:

- **Develop an efficient and universal WF reconstruction algorithm for slope data based on linear equation approach, so that**
- **This algorithm can deal with various irregular pupil shapes in adaptive optics and optical shop testing.**
- **This algorithm is especially useful for large and irregular pupils !**

1. Mathematical Model

Neumann boundary problem

- Wavefront reconstruction from slope data is a Neumann problem of Poisson's equation

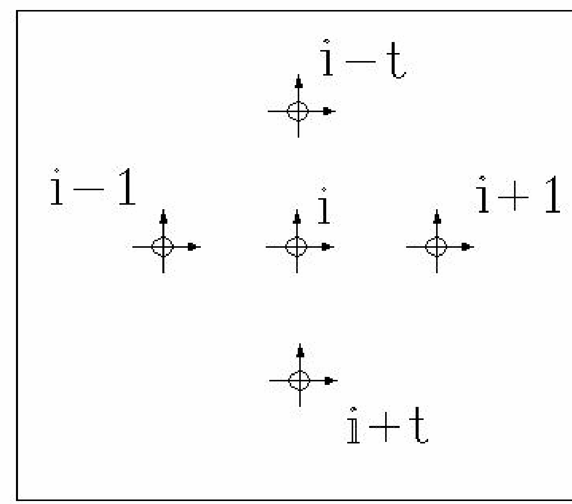
$$\begin{cases} \nabla^2 W = f(y,z), & (y,z) \in \Omega \\ \left.\dfrac{\partial W}{\partial n}\right|_{\partial\Omega} = g(y,z) \end{cases}$$

- The solution to this problem is unique except for an additive constant.

1. Mathematical Model

Postulations

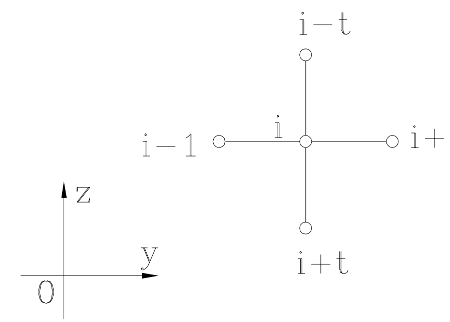
- Suppose we have got slope data (gradients/ derivatives /differences) from WFS, and
- we adopt the Southwell reconstruction geometry (Southwell, JOSA A 70, 1980)



1. Mathematical Model

Zonal integration for Southwell geometry

(D. Su et al, SPIE 2199, pp609-62,1994)



1. If the slopes (gradients) can be linearly interpolated between two consecutive grids, then

$$\frac{\partial W}{\partial y} = \left.\frac{\partial W}{\partial y}\right|_i \left(1 - \frac{y}{a}\right) + \left.\frac{\partial W}{\partial y}\right|_{i+1} \frac{y}{a}$$

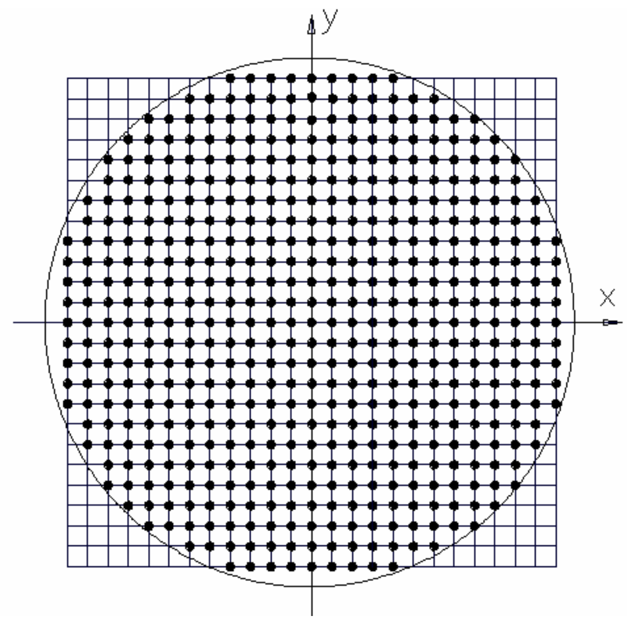
So
$$W_{i+1} - W_i = \frac{1}{2}\left(\left.\frac{\partial W}{\partial y}\right|_i + \left.\frac{\partial W}{\partial y}\right|_{i+1}\right) a$$

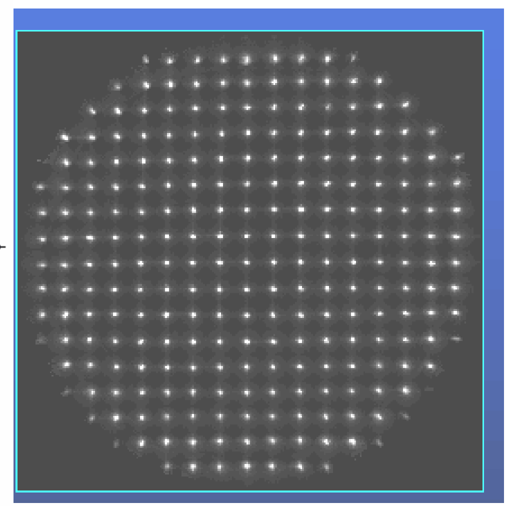
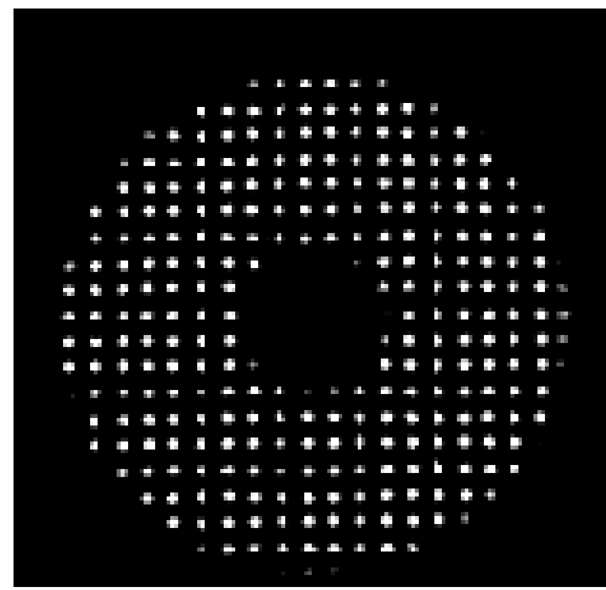
2. For interior point i, we have four phase equations

$$\begin{cases} w_i - w_{i-1} = \frac{1}{2}\left(\left.\frac{\partial W}{\partial y}\right|_i + \left.\frac{\partial W}{\partial y}\right|_{i-1}\right) a \\ w_{i+1} - w_i = \frac{1}{2}\left(\left.\frac{\partial W}{\partial y}\right|_{i+1} + \left.\frac{\partial W}{\partial y}\right|_i\right) a \\ w_{i-t} - w_i = \frac{1}{2}\left(\left.\frac{\partial W}{\partial z}\right|_{i-t} + \left.\frac{\partial W}{\partial z}\right|_i\right) a \\ w_i - w_{i+t} = \frac{1}{2}\left(\left.\frac{\partial W}{\partial z}\right|_i + \left.\frac{\partial W}{\partial z}\right|_{i+t}\right) a \end{cases}$$

2.1 Domain extension technique

- The reconstruction matrix changes with pupil shape and sampling grid indexing mode!



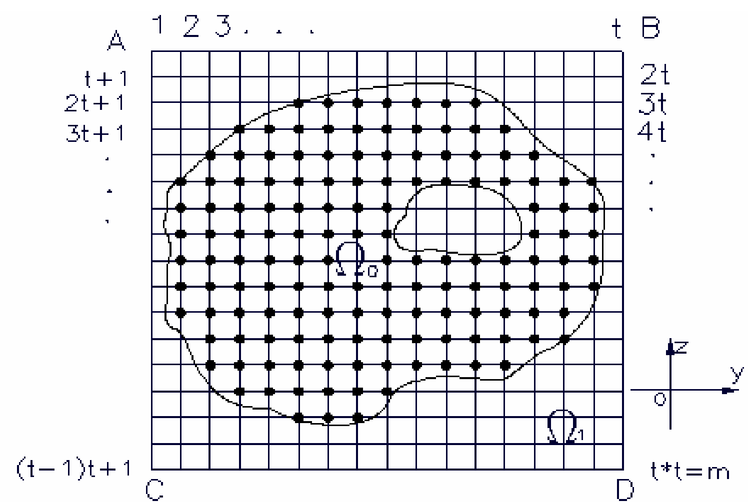



2.1 Domain extension technique

(W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

In developing a generalized wavefront reconstruction algorithm, steps are included:

1. Extending the sampling domain (exit pupil, simply connected domain or multiple connected domains) to a square domain Ω_1 that covers the whole sampling domain Ω_0.
2. Indexing the grids in Ω_1 serially from 1 to m row by row (or colur by column alternatively).
3. Setting the slopes to zero in the additive extended regions $\Omega_1 \backslash \Omega_0$. (i.e. Zero padding!!)



2.2 Generalized WF reconstruction matrix equation (W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

$$
\begin{bmatrix}
-1 & 1 & & & & & & & & & & & & \\
 & -1 & 1 & & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & & \\
 & & & -1 & 1 & & & & & & & & & \\
 & & & & -1 & 1 & & & & & & & & \\
 & & & & & -1 & 1 & & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & & \\
 & & & & & & & -1 & 1 & & & & & \\
 & & & & & & & & \ddots & \ddots & & & & \\
 & & & & & & & & & -1 & 1 & & & \\
 & & & & & & & & & & -1 & 1 & & \\
 & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & -1 & 1 \\
1 & 0 & 0 & \cdots & 0 & -1 & & & & & & & & \\
 & 1 & 0 & 0 & \cdots & 0 & -1 & & & & & & & \\
 & & \ddots & \ddots & \ddots & & \ddots & \ddots & & & & & & \\
 & & & & & & & \ddots & \ddots & \ddots & & \ddots & \ddots & \\
 & & & & & & & & 1 & 0 & 0 & \cdots & 0 & -1 \\
 & & & & & & & & & 1 & 0 & 0 & \cdots & 0 \; -1
\end{bmatrix}
\begin{bmatrix}
w_1 \\ w_2 \\ \vdots \\ w_t \\ w_{t+1} \\ w_{t+2} \\ \vdots \\ w_{2t} \\ \vdots \\ w_{m-2t+1} \\ w_{m-2t+2} \\ \vdots \\ w_{m-t} \\ w_{m-t+1} \\ \vdots \\ w_{m-1} \\ w_m
\end{bmatrix}
=
\begin{bmatrix}
c_{2,1} \\ c_{3,2} \\ \vdots \\ c_{t,t-1} \\ c_{t+2,t+1} \\ \vdots \\ \vdots \\ c_{m,m-1} \\ d_{1,t+1} \\ d_{2,t+2} \\ \vdots \\ d_{t,2t} \\ \vdots \\ d_{i,i+t} \\ \vdots \\ d_{m-t-1,m-1} \\ d_{m-t,m}
\end{bmatrix}
$$

where $c_{i+1,i} = \frac{a}{2}\left(w_{y_{i+1}} + w_{y_i}\right)$ and $d_{i,i+t} = \frac{a}{2}\left(w_{z_i} + w_{z_{i+t}}\right)$

2.2 Generalized WF reconstruction matrix equation (W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

- **Generalized normal equation for WF reconstruction**

$$\mathbf{A}^{T}\mathbf{A}\mathbf{W} = \mathbf{A}^{T}\mathbf{F}$$

where

$$\mathbf{A}^{T}\mathbf{A} = \begin{bmatrix} \mathbf{E}_1 & -\mathbf{I} & & & \\ -\mathbf{I} & \mathbf{E}_2 & -\mathbf{I} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbf{I} & \mathbf{E}_2 & -\mathbf{I} \\ & & & -\mathbf{I} & \mathbf{E}_1 \end{bmatrix}_{\mathbf{m}\times\mathbf{m}}$$

where

$$\mathbf{E}_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -1 \\ & & & -1 & 2 \end{bmatrix}_{t\times t} \quad \mathbf{E}_2 = \begin{bmatrix} 3 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 3 \end{bmatrix}_{t\times t} \quad -\mathbf{I} = \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}_{t\times t}$$

2.2 Generalized WF reconstruction matrix equation (W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

Summary for Normal equation

1. **This is a sparse band matrix!**
2. **No matter how irregular the pupil shapes are, the reconstruction matrices are the same! (except for dimension sizes)!**
3. **$rank(A^TA)=rank(A)=m-1$**
4. **It has 1-dimension solution space!**

3. Solutions

(W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

- Least squares solution with a selected piston value:
  → manually set a zero point for WF

$$W = (A^T A + \mu I)^{-1} A^T F$$

$$\mu = diag(0,0,\cdots 0,10^{30},0,\cdots,0)$$

- Least squares solution with minimum norm (LSMN):
  → Select a zero point so that the LS solution satisfies

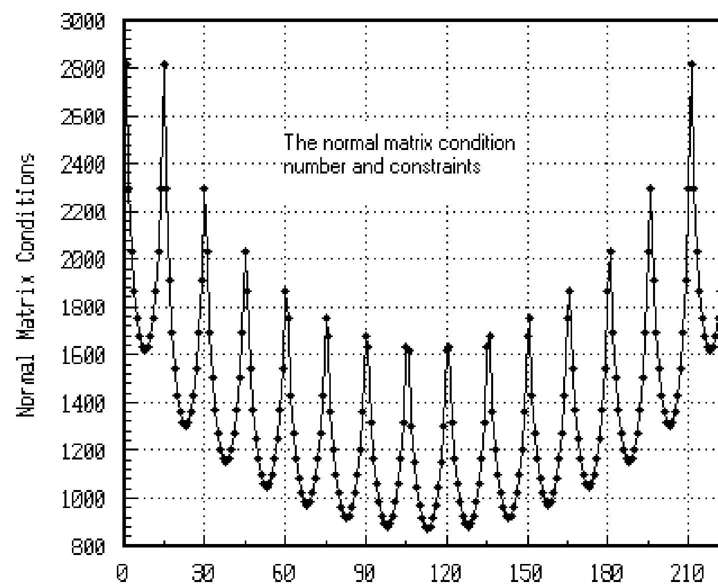
$$\sum_{i=1}^{m} {w_i}^2 = \text{minimum}$$

3. Solutions

(W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

- **Which point is a good "zero point" for a wave front?**

1. The best constraint point is located at the center of the testing pupil,
2. at which the condition number of normal equation is the smallest.



Definitions

Condition number of normal matrix:

$$\mathbf{cond}(\mathbf{A}^T\mathbf{A}) = \mathbf{lub}_2\left[(\mathbf{A}^T\mathbf{A})^{-1}\right]\mathbf{lub}_2\left(\mathbf{A}^T\mathbf{A}\right)$$

and matrix norm:

$$\mathbf{lub}_2(\mathbf{A}) = \max_{\mathbf{x}\neq\mathbf{o}}\sqrt{\frac{\mathbf{X}^T\mathbf{A}^T\mathbf{A}\mathbf{X}}{\mathbf{X}^T\mathbf{X}}} = \sqrt{\rho(\mathbf{A}^T\mathbf{A})}$$

3. Solutions

(W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

How to obtain the Minimum-Norm Least-Squares solution (MNLS)?

For most cases in AO and Optical tests, We are looking for the Mini-Norm LS solutions. MNLS can be obtained in following means:

1. Select a "free " zero point for the wavefront, and in the LS sense we look for the solution for

$$\sum_{i=1}^{m} w_i^{\ 2} = 0$$

2. Damping least squares solution converges to MNLS solution, when the damping factor is very small.
3. Iterative method, when the initial values are zero.
4. Singular Value Decomposition method (SVD)

3. Solutions

How to solve the normal equation set?

1. Iterative methods
2. Singular Value Decomposition method (SVD)
3. Gaussian elimination method
4. Cholesky decomposition method
5. ....

3. Solutions (W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

Numerical methods for solving the Normal equation

→ Iterative methods

1. Jacobi method (D. Su at al 2000)

$$\mathbf{w}_{\mathbf{i}} = \frac{1}{4}\left(\mathbf{w}_{\mathbf{i-t}} + \mathbf{w}_{\mathbf{i-1}} + \mathbf{w}_{\mathbf{i+1}} + \mathbf{w}_{\mathbf{i+t}}\right) + \frac{\mathbf{a}}{8}\left(\left.\frac{\partial \mathbf{W}}{\partial \mathbf{y}}\right|_{\mathbf{i-1}} + \left.\frac{\partial \mathbf{W}}{\partial \mathbf{z}}\right|_{\mathbf{i+t}} - \left.\frac{\partial \mathbf{W}}{\partial \mathbf{z}}\right|_{\mathbf{i-t}} - \left.\frac{\partial \mathbf{W}}{\partial \mathbf{y}}\right|_{\mathbf{i+1}}\right)$$

$$\mathbf{w}_{1} = \frac{1}{2}\left(\mathbf{w}_{2} + \mathbf{w}_{\mathbf{t+1}}\right) + \frac{\mathbf{a}}{4}\left(\left.\frac{\partial \mathbf{W}}{\partial \mathbf{z}}\right|_{1} + \left.\frac{\partial \mathbf{W}}{\partial \mathbf{z}}\right|_{\mathbf{t+1}} - \left.\frac{\partial \mathbf{W}}{\partial \mathbf{y}}\right|_{2} - \left.\frac{\partial \mathbf{W}}{\partial \mathbf{y}}\right|_{1}\right)$$

2. Gauss -Seidel method

3. SOR method → most efficient :

   Optimal relaxation factor is 1.881, corresponding iteration times for convergence is 111

3. Solutions (W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

Numerical methods for solving the Normal equation

→Cholesky Decomposition Method

The normal equation must be positive $\mathbf{A}^T\mathbf{A} > 0$

1. Decomposition ($m^3/6$ +m square-root computations)

$$A^T A = [a_{ij}]_{m\times m} \qquad \mathbf{B} = [\mathbf{b}_{ij}]_{\mathbf{m\times m}} = \begin{cases} 0 & i < j \\ \mathbf{b}_{ij} & i \ge j \end{cases} \qquad a_{ij} = \sum_{k=1}^{i} b_{ik} b_{jk}, \quad (1 \le i \le j \le m)$$

Then the normal equation $\mathbf{BB^TW=Y}$

Let $\mathbf{U=B^TW}$, $\mathbf{U}=(\mathbf{u}_1,\mathbf{u}_2,\ldots\mathbf{u}_m)^T$, so $\mathbf{BU=Y}$

2. Substitution (m^2 computations)

$$u_1 = y_1/b_{11}$$

$$u_i = (y_i - \sum_{j=1}^{i-1} b_{ij} y_j) \Big/ b_{ii}, \qquad i = 2,3,\cdots,m$$

$$w_m = u_m/b_{m,m}$$

$$w_i = \left(u_i - \sum_{j=i+1}^{m} b_{ji} w_j \right) \Big/ b_{ii}, i = m-1, m-2, \cdots, 1$$

3. Solutions

(W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

The memory storage problem in solving the normal equation

- In stead of saving it in memory space, we express the matrix coefficients as a function a(i, j), which has only 5 different values.

if $i \ge j$

$$a(i,j)=\begin{cases} 4, & if\,(i,j)\in\Omega_4 \\ 3, & else\,if\,(i,j)\in\Omega_3 \\ 2, & else\ \ if\,(i,j)\in\Omega_2 \\ -1, & else\ \ if\,(i,j)\in\Omega_{-1} \\ 0, & else \end{cases}$$

else

$$a(i,j) = a(j,i)$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{E}_1 & -\mathbf{I} & & & \\ -\mathbf{I} & \mathbf{E}_2 & -\mathbf{I} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbf{I} & \mathbf{E}_2 & -\mathbf{I} \\ & & & -\mathbf{I} & \mathbf{E}_1 \end{bmatrix}_{m\times m}$$

$$\mathbf{E}_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -1 \\ & & & -1 & 2 \end{bmatrix}_{t\times t} \quad \mathbf{E}_2 = \begin{bmatrix} 3 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 3 \end{bmatrix}_{t\times t} \quad -\mathbf{I} = \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}_{t\times t}$$

- Therefore, we only need mx(t+1) elements of memory space for Cholesky decomposition.

4. Error analysis (W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

1. Discretization errors of the normal equation

For Interior grids $\sim a^4$, a is the grid interval.

$$-\mathbf{w}'_{i+1} - \mathbf{w}'_{i+t} + 4\mathbf{w}'_{i} - \mathbf{w}'_{i-1} - \mathbf{w}'_{i-t} = -\mathbf{a}^2 \nabla \mathbf{n}_i - \frac{\mathbf{a}^4}{12}\left(\left.\frac{\partial^4 \mathbf{W}'}{\partial \mathbf{y}^4}\right|_i + \left.\frac{\partial^4 \mathbf{W}'}{\partial \mathbf{z}^4}\right|_i \right) + O(\mathbf{a}^6)$$

For Boundary grids $\sim a^3$

$$\mathbf{w}_i - \mathbf{w}_{i+t} = \mathbf{a}\left.\frac{\partial \mathbf{W}}{\partial \mathbf{z}}\right|_{i+t}^{i} + \frac{\mathbf{a}^3}{24}\left.\frac{\partial^3 \mathbf{W}}{\partial \mathbf{z}^3}\right|_{i+t}^{i} + O(\mathbf{a}^5)$$

So, the less complexity the pupil shape is, the higher precision the algorithm can achieve.

→ Remove the Boundary effect , see Roddier at al, Appl Opt, 1987.

4. Error analysis (W. Zou and Z. Zhang, *Appl. Opt.*, Vol. 39, No.2, Jan.2000)

2. Error propagation

$$\mathbf{A}^{T}\mathbf{A}\mathbf{W}' = \frac{\mathbf{a}^{4}}{24}\left(\mathbf{M}_{3} - 2\mathbf{M}_{4}\right) + \mathbf{a}\mathbf{A}^{T}\mathbf{N}'$$

3. Wavefront Error estimate

$$\sigma_{W'} \le \frac{a^{4}}{24\cdot t}\frac{cond\left(A^{T}A\right)}{\mathrm{lub}_{2}(A)^{2}}\left\|M_{3} - 2M_{4}\right\|_{2} + \frac{a\beta\sqrt{k}}{f_{0}\cdot t}\frac{\left[cond\left(A^{T}A\right)\right]^{\frac{1}{2}}}{\mathrm{lub}_{2}(A)}\sigma_{CCD}$$

Discretization error — Centroiding error

4. Error estimate for a specific S-H system

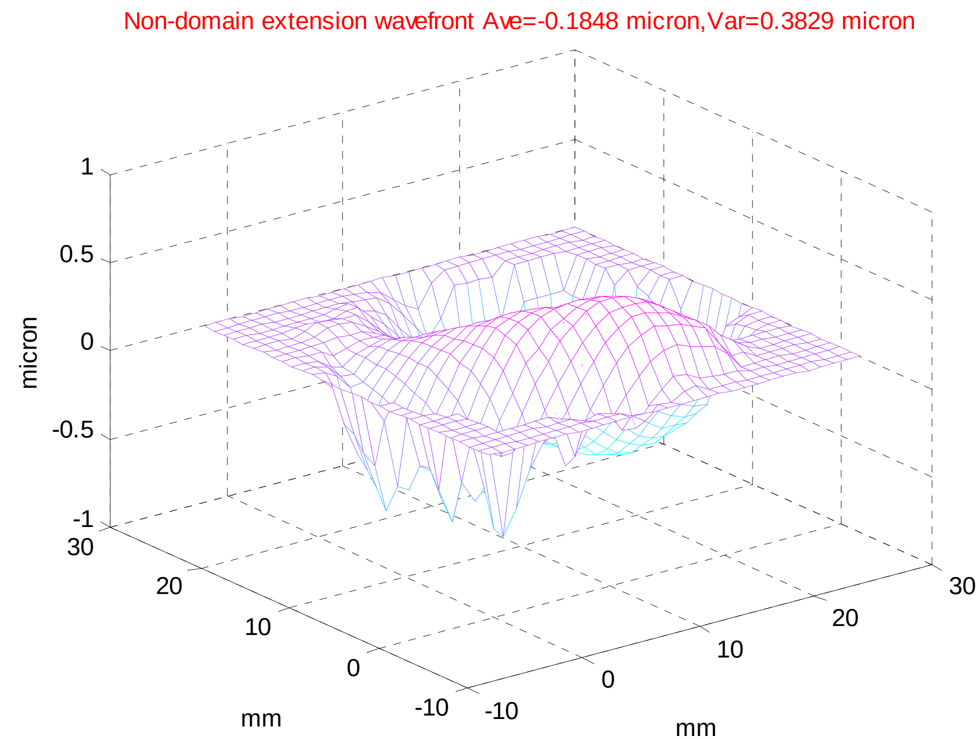
(neglect discretization errors, and, m=txt=15x15, **a**=2mm, **f0=**184mm, β=7.5)

$$\sigma_{W'} \le 17.752\frac{\beta}{f_{0}}\sigma_{CCD} = 0.724\sigma_{CCD}$$

5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

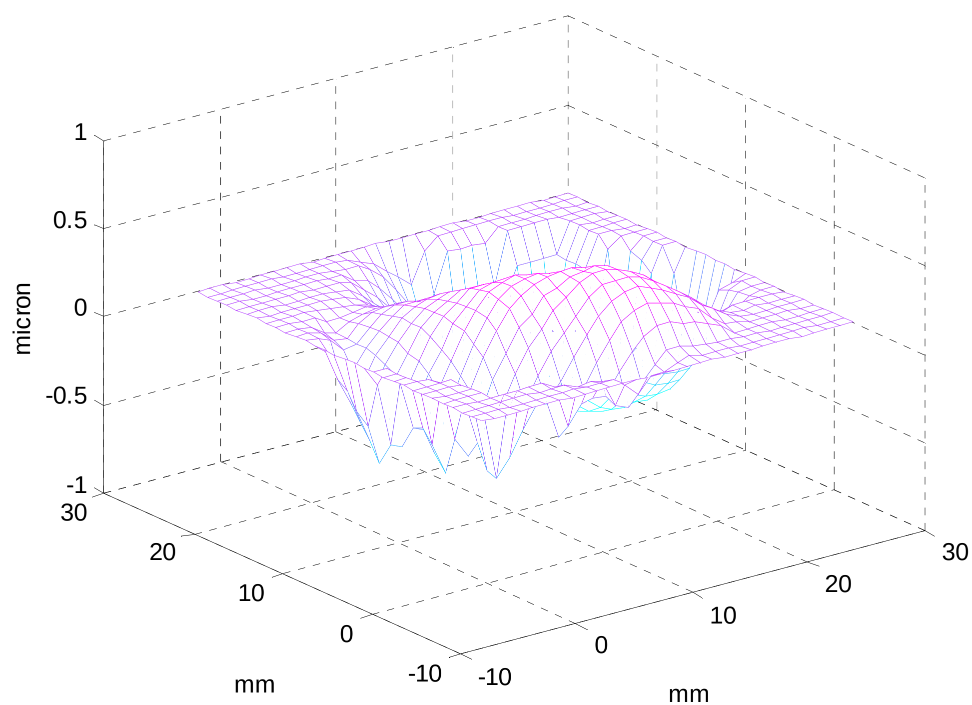
1. **Original wavefront**



5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

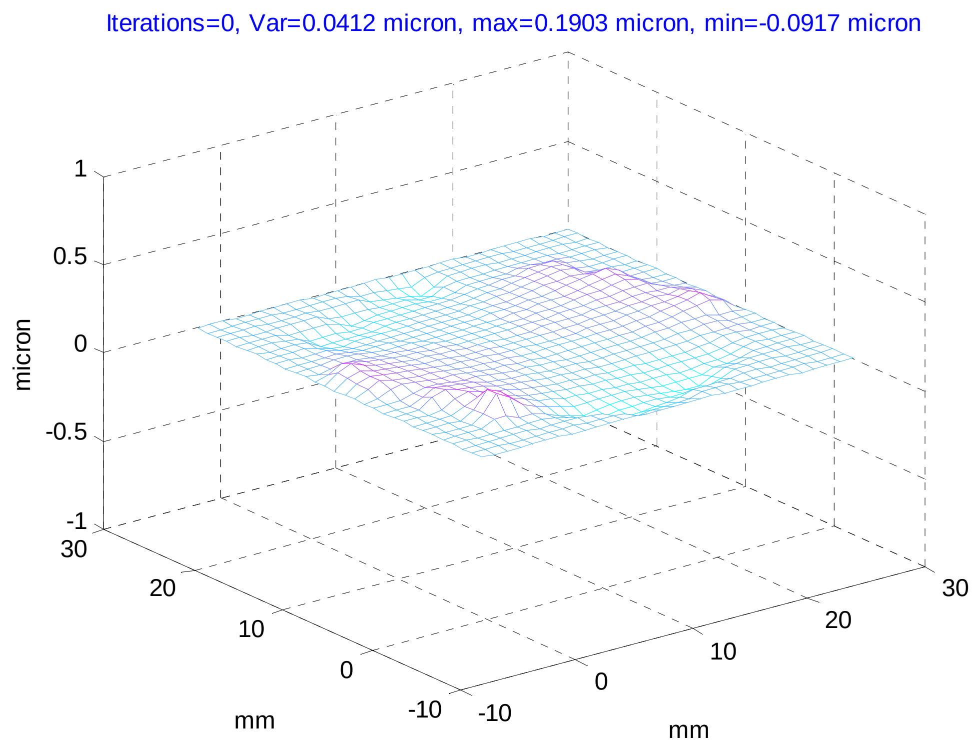
2. **The reconstructed wavefront with domain extension technique**



5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

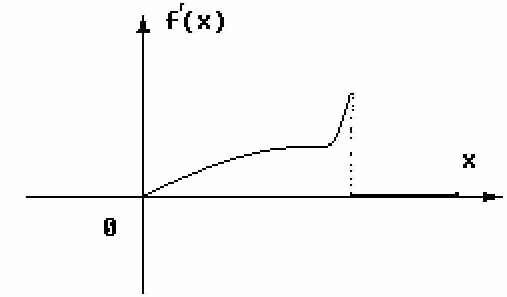
We have λ /3 - λ /2 P-V deviation error (λ =632.8nm) induced between the two wave fronts!!!☹ Why???



5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

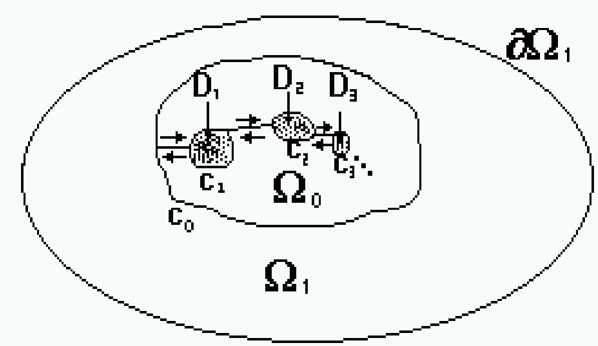
Compatibility of Domain extension



Since the extended slope data does not satisfy the derivative continuity condition of Neumann problem, so the domain extension is not strictly compatible.

Conclusion:

1. Zero padding of slopes outside the pupil will introduce deviation errors!
2. We need to extrapolate the slope data outside the pupil instead of zero padding!



5. Deviation error removal

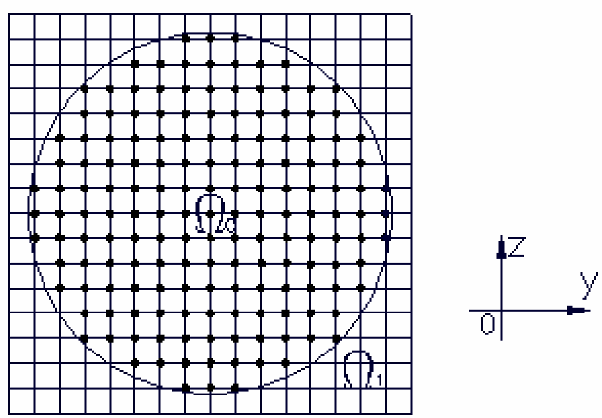
(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

- **We employ an iterative linear equation approach to remove the deviation error induced by domain extension in the Generalized wavefront reconstruction algorithm!**

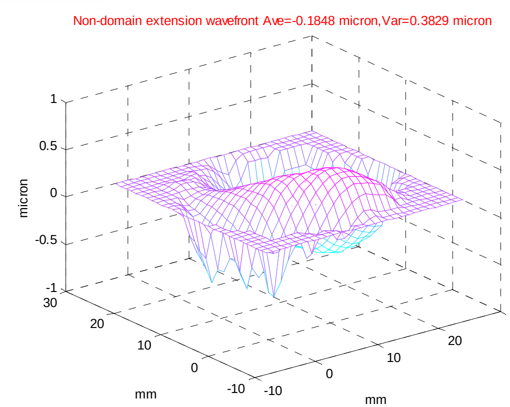
5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

Example 1. Optical pupil without obstruction

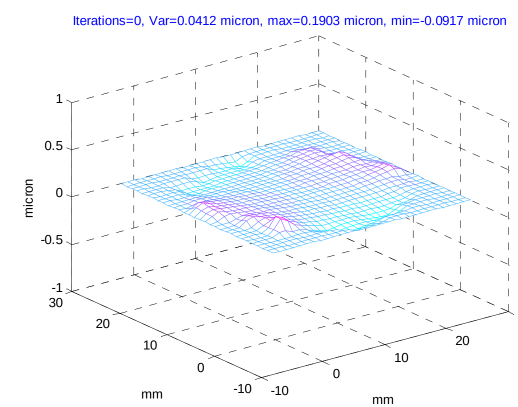


A pupil without obstruction



a) Original wavefront

i=0 ,Rms=0.0412 μm

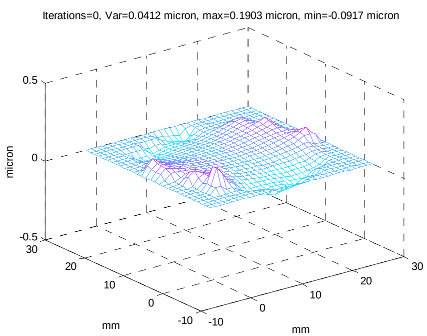


b) Deviation error of the reconstructed WF from the original

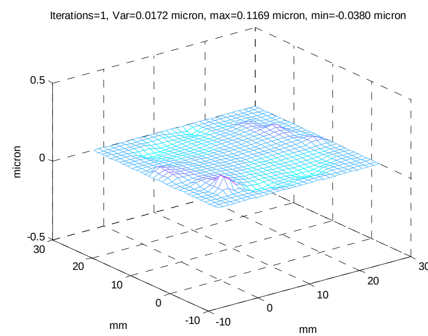
5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

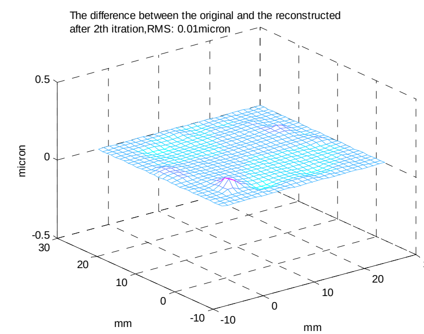
Example 1. Algorithm Convergence process......



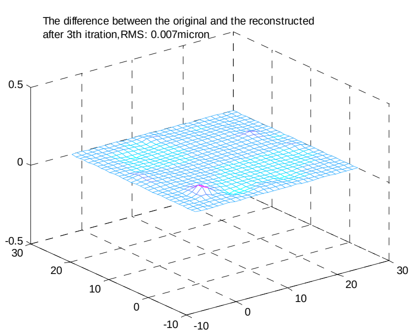
i=0, Rms=0.04 μ m



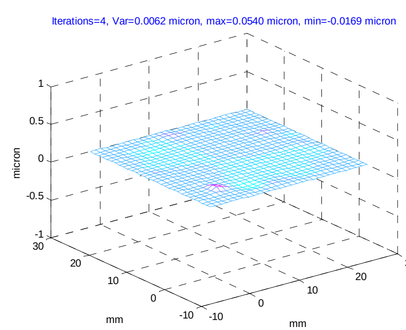
i=1, Rms=0.02 μ m



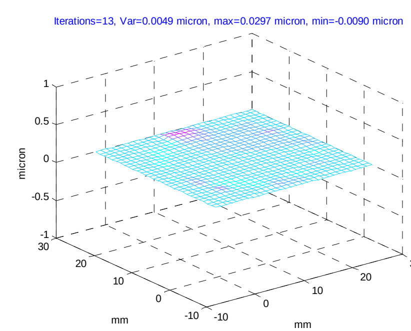
i=2, Rms=0.01 μ m



i=3, Rms=0.007 μm



i=4, Rms=0.006 μm



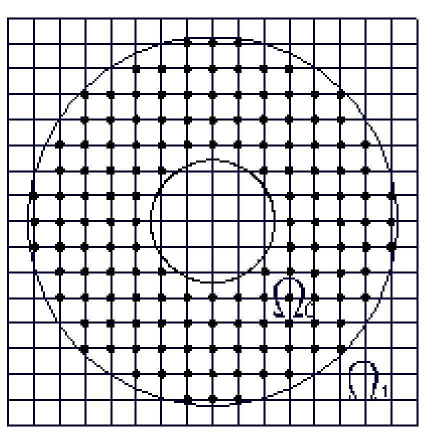
i=13, Rms=0.005 μ m

5. Deviation error removal

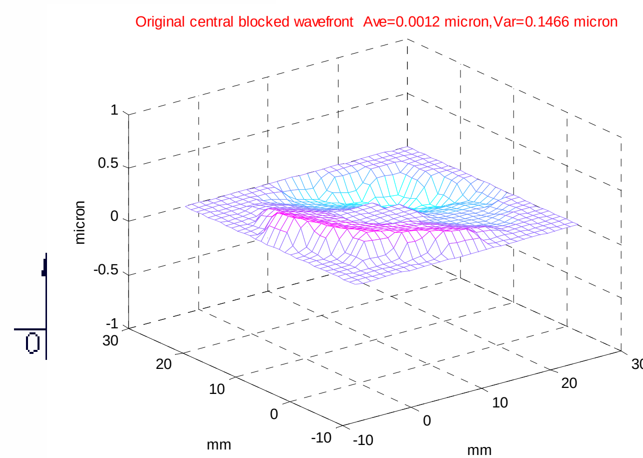
(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

Example 2. Optical pupil with central obstruction

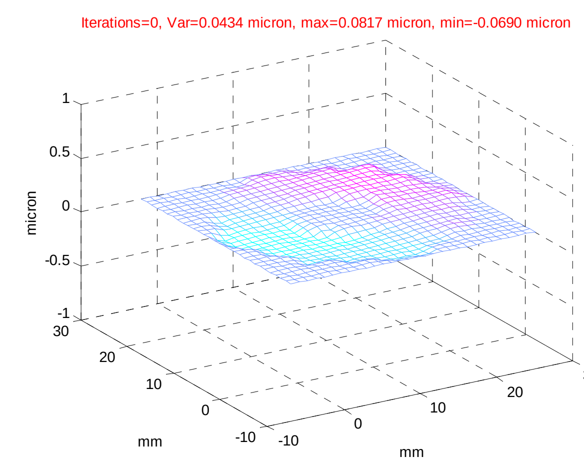
i=0, Rms=0.043 μm



A pupil with central obstruction



a) Original wavefront

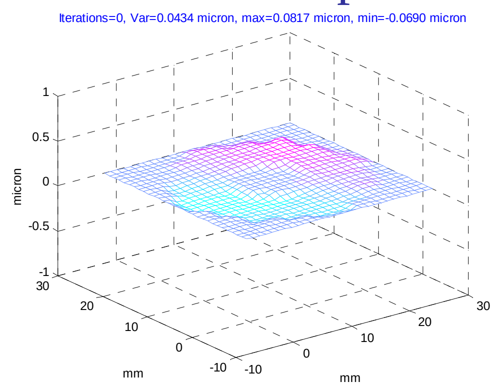


b) Deviation error of the reconstructed WF from the original

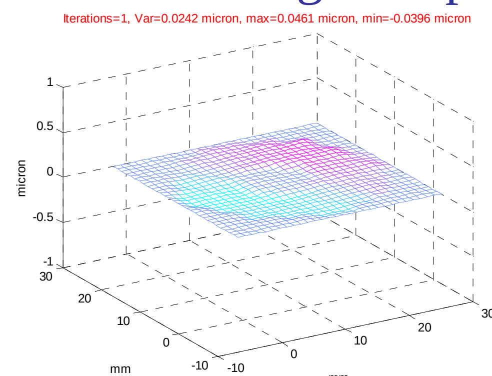
5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

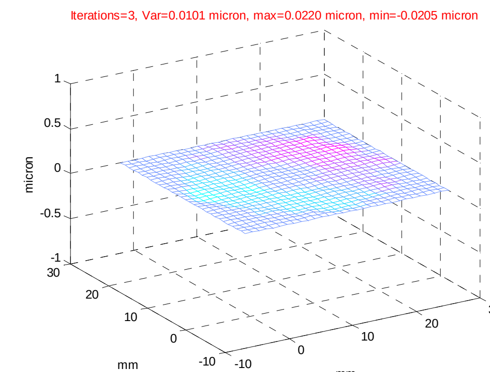
Example 2. Algorithm Convergence process……



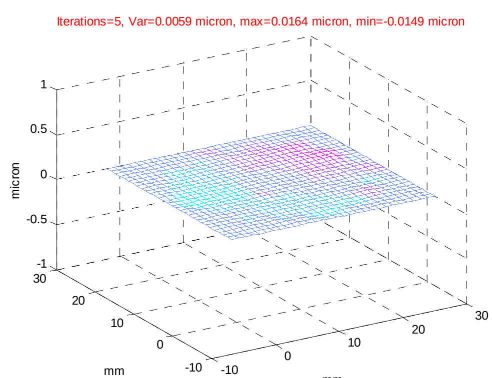
i=0, Rms=0.043 μm



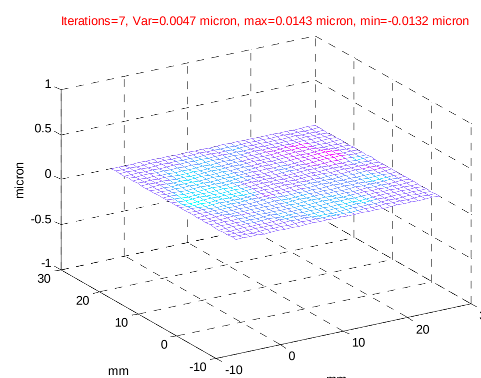
i=1, Rms=0.024 μm



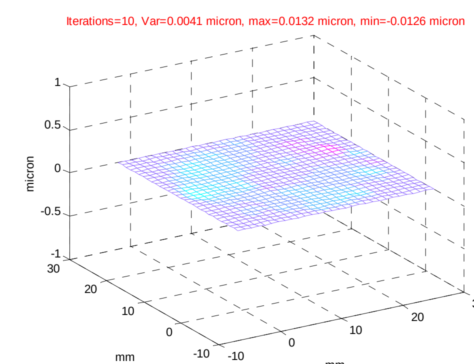
i=3, Rms=0.01 μm



i=5, Rms=0.006 μm



i=7, Rms=0.005 μm



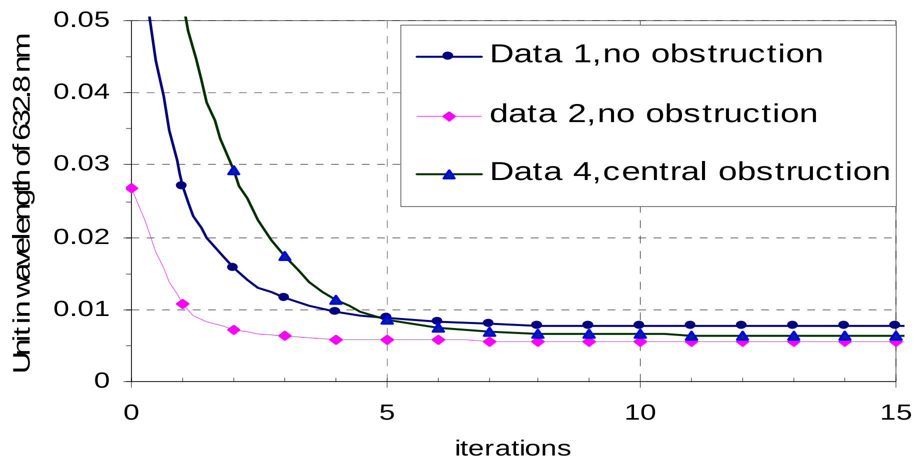
i=10, Rms=0.004 μm

5. Deviation error removal

(W. Zou and J Rolland, *Submitted to JOSA A, 2003*)

The deviation achieved with this algorithm is less than λ/150~λ/200 for λ=632.8nm, which is excellent for most optical testing and AO systems.

Computation time for our PC ~1sec ~2 sec



Deviation errors vs Iteration times

Slope data is from a Shack-Hartmann sensor

6. Summary

- **Fourier Transform- based algorithm**

  1. Klaus Freischlad and Chris Koliopoulos did a pioneer work for generalized algorithm techniques (Freischlad and Koliopoulos, JOSA A Vol. 3, Zo. 11, 1986)
  2. Roddier et al use a Gerchberg-type iterative algorithm to extrapolate interferogram outside the pupil and obtained a perfect domain extension.

- **Linear equation set-based algorithm**

  1. W. Zou proposed a generalized algorithm with zero padding the slope data outside the domain (W. Zou at al, Appl. Opt., Vol. 39, No.2, Jan.2000)
  2. W. Zou and J. Roland improved their previous technique by extrapolating the slope data outside the pupil with iterative linear equation approach (Submitted to JOSA A in 2003 and patent is in filing...)

7. References

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