

Sparse Matrix Techniques for MCAO

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Wavefront Estimation from Idealized WFS Data

- Open-loop WFS: $s = \mathcal{G}\varphi + \eta$
 - η and φ random with known statistics $C_\eta = \langle \eta \eta^T \rangle$ and $C_\varphi = \langle \varphi \varphi^T \rangle$
- $\Rightarrow \hat{\varphi} = \mathcal{G}^\dagger s$ (noise-weighted pseudo-inverse)

$$\hat{\varphi} = \arg \min_{\varphi} J(\varphi)$$

$$J(\varphi) = \frac{1}{2} \left[\|\eta\|_{C_\eta^{-1}}^2 + \|\varphi\|_{C_\varphi^{-1}}^2 \right] = \frac{1}{2} \left[\langle \eta, C_\eta^{-1} \eta \rangle + \langle \varphi, C_\varphi^{-1} \varphi \rangle \right]$$

$$\Rightarrow \hat{\varphi} = \mathcal{G}^\dagger s \quad \text{with} \quad \mathcal{G}^\dagger = (\mathcal{G}^T C_\eta^{-1} \mathcal{G} + C_\varphi^{-1})^{-1} \mathcal{G}^T C_\eta^{-1}$$

- Block-layered structure for MCAO
- $(\text{SNR})^2 = \langle \|\mathcal{G}\varphi\|^2 \rangle / \langle \|\eta\|^2 \rangle = \langle \|\mathcal{G}\varphi\|^2 \rangle / (2 n_z \sigma^2)$

Key Approximations to Inverse Turbulence Covariance

Approximation #1

- C_φ is block diagonal (different layers are statistically independent)
 - For each turbulent layer l , C_{φ_l} is BTTB (phase structure function)
 - Approximate BTTB matrix by BCCB matrix (diagonalized by DFT)
 - $C_{\varphi_l} \simeq \text{BCCB}(\tau_l) = \mathcal{F}^{-1} \text{diag}(\hat{\tau}_l) \mathcal{F}$ where $\hat{\tau}_l = \mathcal{F} \tau_l$ (eigenvalues)
 - $\hat{\tau}_l = c_l |\kappa|^{-11/3}$ (Kolmogorov PSD)
- $\Rightarrow C_{\varphi_l}^{-1} \simeq \text{BCCB}(\Lambda_l) = \mathcal{F}^{-1} \text{diag}(\lambda_l) \mathcal{F}, \quad \lambda_l = \hat{\tau}_l^{-1} = c_l^{-1} |\kappa|^{11/3}$

Key Approximations to Inverse Turbulence Covariance

Approximation #2

- $\text{BCCB}(\Lambda_l)$ is a full matrix.
- Entries Λ_l rapidly decay to small values
- Approximate $\text{BCCB}(\Lambda_l)$ by \mathcal{S}_l^2 where $\mathcal{S}_l = c_l' \mathcal{S}$ and \mathcal{S} is the discrete Laplacian matrix with periodic boundary conditions

$$\mathcal{S} = \text{BCCB}(v) = \mathcal{F}^{-1} \text{diag}(\hat{v}) \mathcal{F} \quad (\text{sparse BCCB})$$

$$\hat{v} = 4 \left[\sin^2(\pi \kappa_x \Delta x) / \Delta x^2 + \sin^2(\pi \kappa_y \Delta y) / \Delta y^2 \right] \xrightarrow{\Delta x, \Delta y \rightarrow 0} 4\pi^2 |\kappa|^2$$

- $\mathcal{S} = \mathcal{S}_y \otimes I_x + I_y \otimes \mathcal{S}_x$ where \mathcal{S}_y and \mathcal{S}_x are 1D versions

Key Approximations to Inverse Turbulence Covariance

Sparsity Patterns in 1D

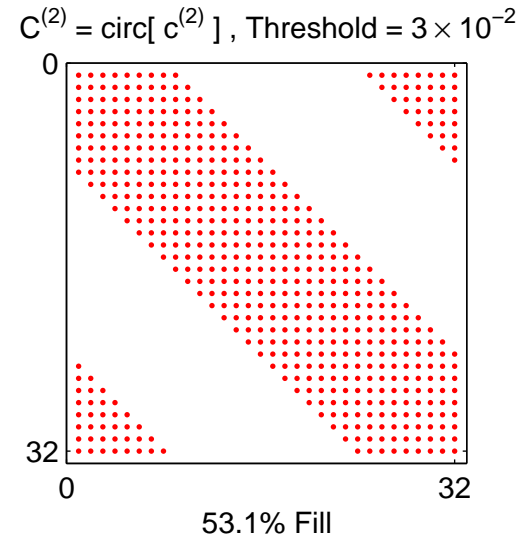
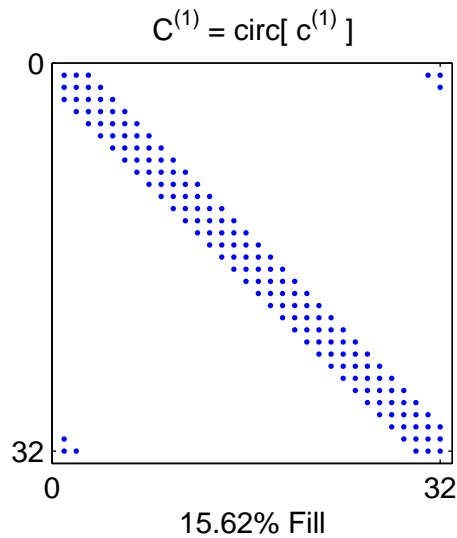
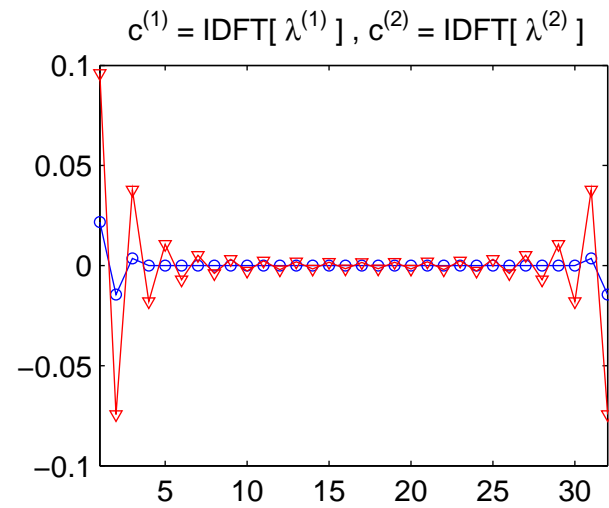
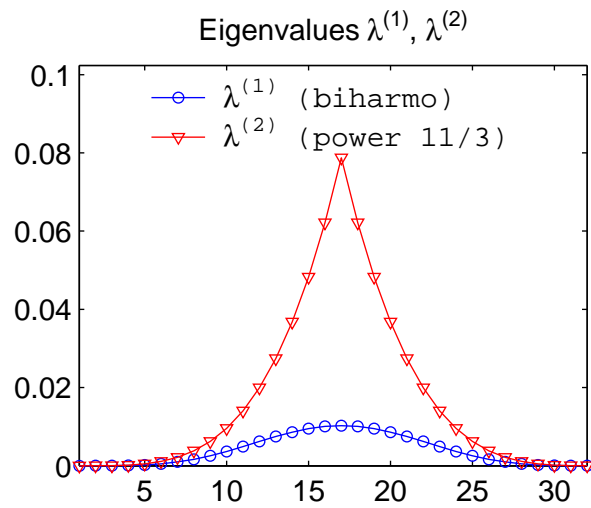
$$[\mathcal{S} u]_i = (-u_{i-1} + 2u_i - u_{i+1}) / \Delta x^2$$

$$\mathcal{S} = \text{circ}(v), \quad v = (2, -1, 0, \dots, 0, -1)^T / \Delta x^2 = (2e_0 - e_1 - e_{n-1}) / \Delta x^2$$

$$\Rightarrow \mathcal{S} = \mathcal{F}^{-1} \text{diag}(\hat{v}) \mathcal{F}, \quad \hat{v} = \mathcal{F}v = (2\hat{e}_0 - \hat{e}_1 - \hat{e}_{n-1}) / \Delta x^2$$

$$\hat{v} = 4 \sin^2(\pi \kappa \Delta x) / \Delta x^2 \quad (\text{eigenvalues}) \quad \xrightarrow{\Delta x \rightarrow 0} 4\pi^2 \kappa^2$$

Key Approximations to Inverse Turbulence Covariance



MCAO Minimum Variance Reconstructor

Problem: find optimal actuator commands \hat{a} minimizing MCAO wide-field error metric \mathcal{W}

$$\hat{a} = \mathcal{R}s$$

$$\mathcal{R} = \arg \min_R J(R)$$

$$J(R) = \langle \|\epsilon\|_{\mathcal{W}}^2 \rangle = \langle \epsilon^T \mathcal{W} \epsilon \rangle$$

$$\epsilon = H^a \hat{a} - H^\varphi \varphi \quad (\text{aperture-plane residual phase})$$

$$\Rightarrow \mathcal{R} = \underbrace{H^{a\dagger}}_{\text{Fitting}} H^\varphi \underbrace{\mathcal{G}^\dagger}_{\text{Estimation}}$$

$$\mathcal{G}^\dagger = \left(\underbrace{\mathcal{G}^T C_\eta^{-1} \mathcal{G}}_{\text{Sparse+low-rank (LGS)}} + \underbrace{C_\varphi^{-1}}_{\text{Sparse approx}} \right)^{-1} \mathcal{G}^T C_\eta^{-1}$$

$$H^{a\dagger} = \left(H^{aT} \mathcal{W} H^a + \text{low-rank} \right)^{-1} H^{aT} \mathcal{W}$$

Multigrid (MG) Methods

- Can sometimes be used as “stand-alone” system solvers.
- Can be used as preconditioners.
- Rely on multiple scales (grid sizes) inherent in certain problems.
 - Need “smoother” which damps out high-frequency components of error on fine grids.
 - Classical Gauss-Seidel iteration works well for Laplace’s equation.
 - Remaining low frequency error is well-represented on coarser grids.
- Are recursive versions of the following 2-grid scheme.

2-Grid Scheme

$$x_h \leftarrow S(x_h, y_h, \dots)$$

$$r_h \leftarrow A_h x_h - y_h$$

Restrict $r_H \leftarrow I_h^H r_h$

Interpolate $e_h \leftarrow I_H^h e_H$

Solve $A_H e_H = r_H$

$$x_h \leftarrow x_h + e_h$$

$$x_h \leftarrow S(x_h, y_h, \dots)$$

- $S(v, w, \dots)$ denotes application of **smoother** to solve $Ax = w$ with initial guess $x = v$.
- To obtain MG V-cycle, apply 2-grid scheme recursively. Carry out **Solve** step with (e_H, r_H) in place of (x_h, y_h) .

Multigrid (MG) Methods

- Inter-grid transfers (restriction, or up-binning, and interpolation, or down-binning) are cheap.
- Cost is typically dominated by smoother application on finest grid.
- Choice of smoother is problem-dependent.
 - Block (i.e., layer-oriented) symmetric Gauss-Seidel (B-SGS) works well for MCAO estimation step.
 - FFT-based modified Richardson iteration works well for Ex-AO estimation.

Block Gauss-Seidel Smoother

Based on block $L + D + U$ splitting

$$A = \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ A_{21} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A_{n1} & \dots & A_{n,n-1} & 0 \end{bmatrix}}_L + \underbrace{\begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_{nn} \end{bmatrix}}_D \\
 + \underbrace{\begin{bmatrix} 0 & A_{12} & \dots & A_{1n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{n-1,n} \\ 0 & \dots & 0 & 0 \end{bmatrix}}_U$$

Block Gauss-Seidel Smoother

$Ax = b$ is equivalent to $(L + D)x = b - Ux$. This motivates the block forward iteration

$$(L + D)x_{k+1} = b - Ux_k, \quad k = 0, 1, \dots$$

Similarly, we obtain the block backward iteration

$$(D + U)x_{k+1} = b - Lx_k, \quad k = 0, 1, \dots$$

Block symmetric Gauss-Seidel (B-SGS) is obtained by interweaving forward and backward iterations.

Efficient PCG Solver

Estimation Step

- MG preconditioned CG
- Block SGS smoother: requires only inversion of diagonal blocks. Implemented using reordering + full Cholesky factorization.

Fitting Step

- Incomplete Cholesky preconditioned CG. Incomplete Cholesky applied to full sparse matrix without reordering.

Preliminary Results

- ★ Algorithm tested against conventional matrix multiply reconstructors on 8m class problems with $\sim 7 \cdot 10^3$ degrees of freedom
- ★ 32m class problems with $\sim 7 \cdot 10^4$ degrees of freedom solvable in Matlab with 2-3Gb memory
- ★ Convergence obtained in 2-10 iterations
 - Convergence rate a strong function of WFS noise level
 - Weak function of problem dimensionality, NGS vs. LGS MCAO

Sample MCAO Problem Dimensionality

Aperture diameter (m)	8	16	32
WFS measurements	2240	8560	33320
Turbulence phase points	7270	21226	70838
DM actuators	789	2417	8449

Sample MCAO Problem Dimensionality

Estimation matrix to be inverted. 6 layers, 5 NGS's. D=16m. Phase screens 128×128 .

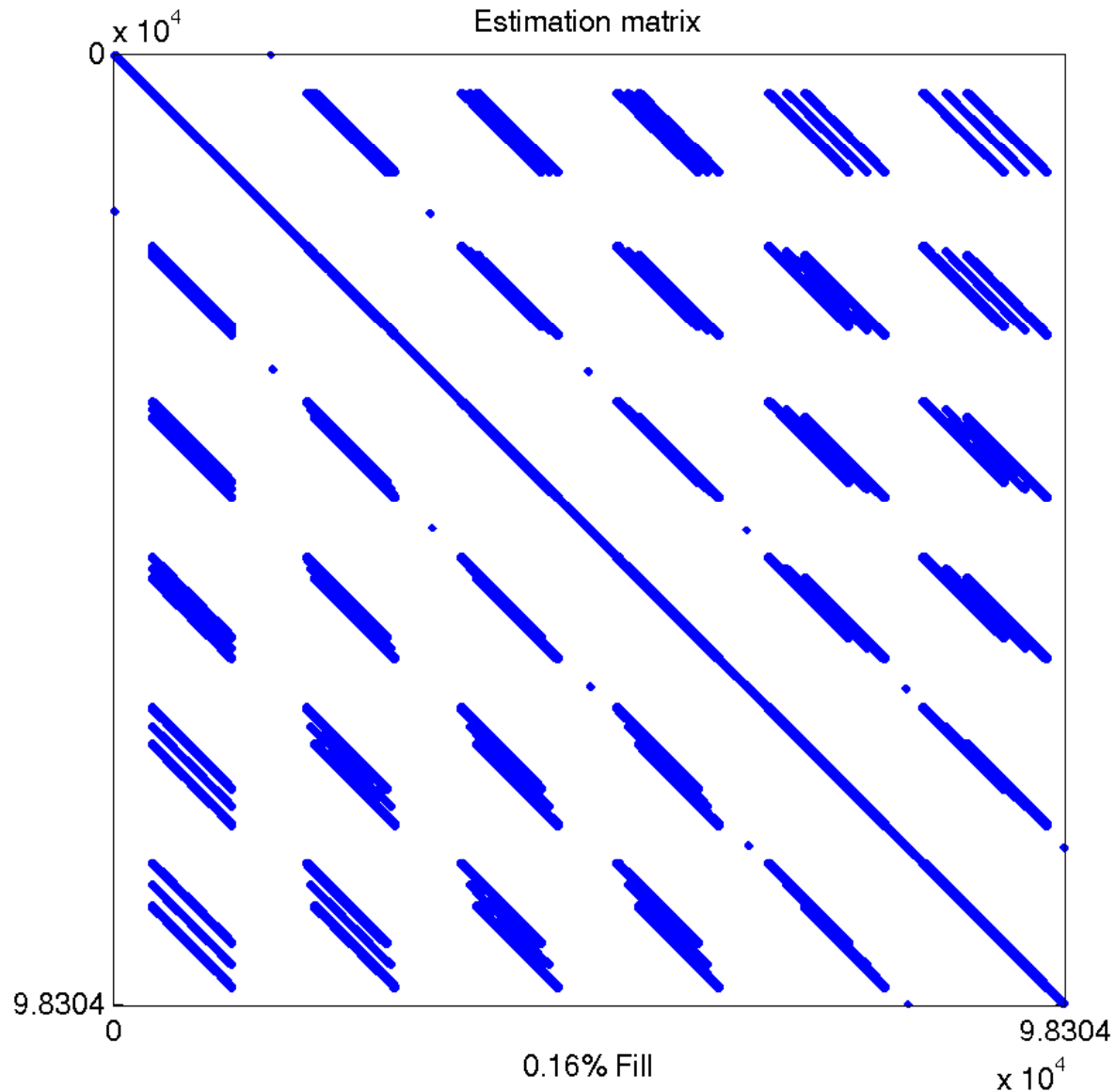


Fig.1

Top layer diagonal block of estimation matrix. $D=16m$. Phase screens 128×128 .

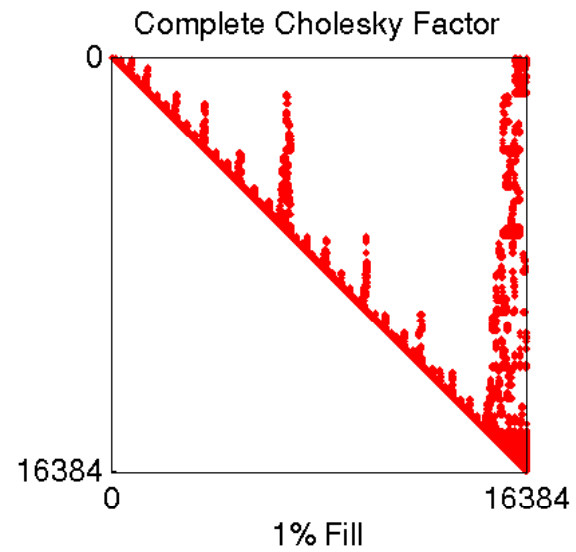
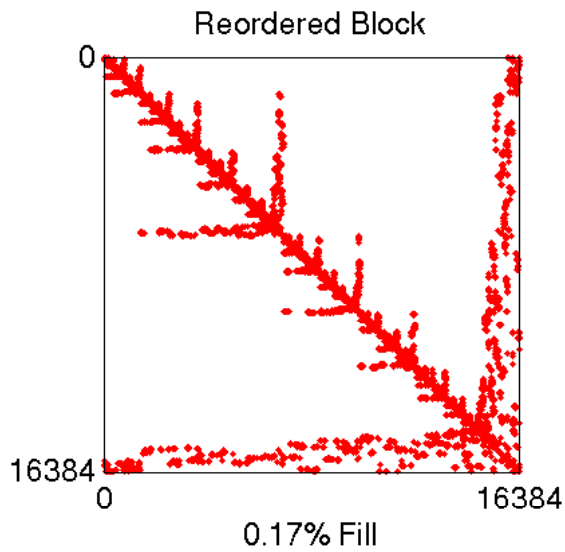
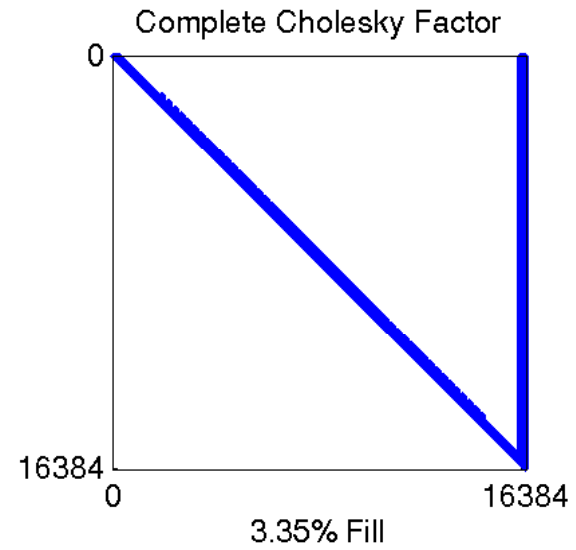
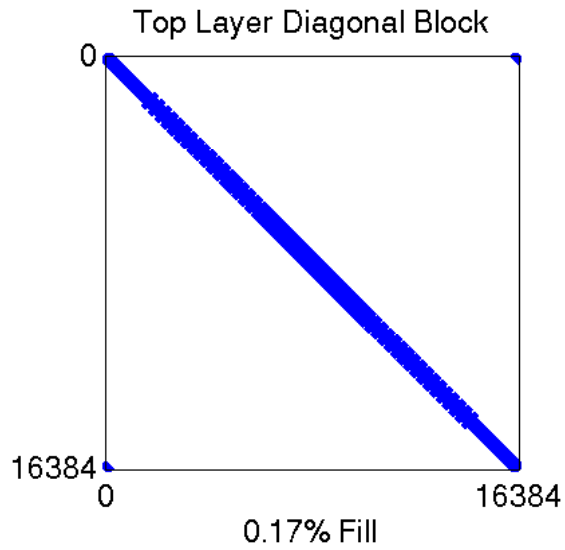


Fig.2

Fitting matrix (3 DM's, 932 actuators/DM). D=16m.

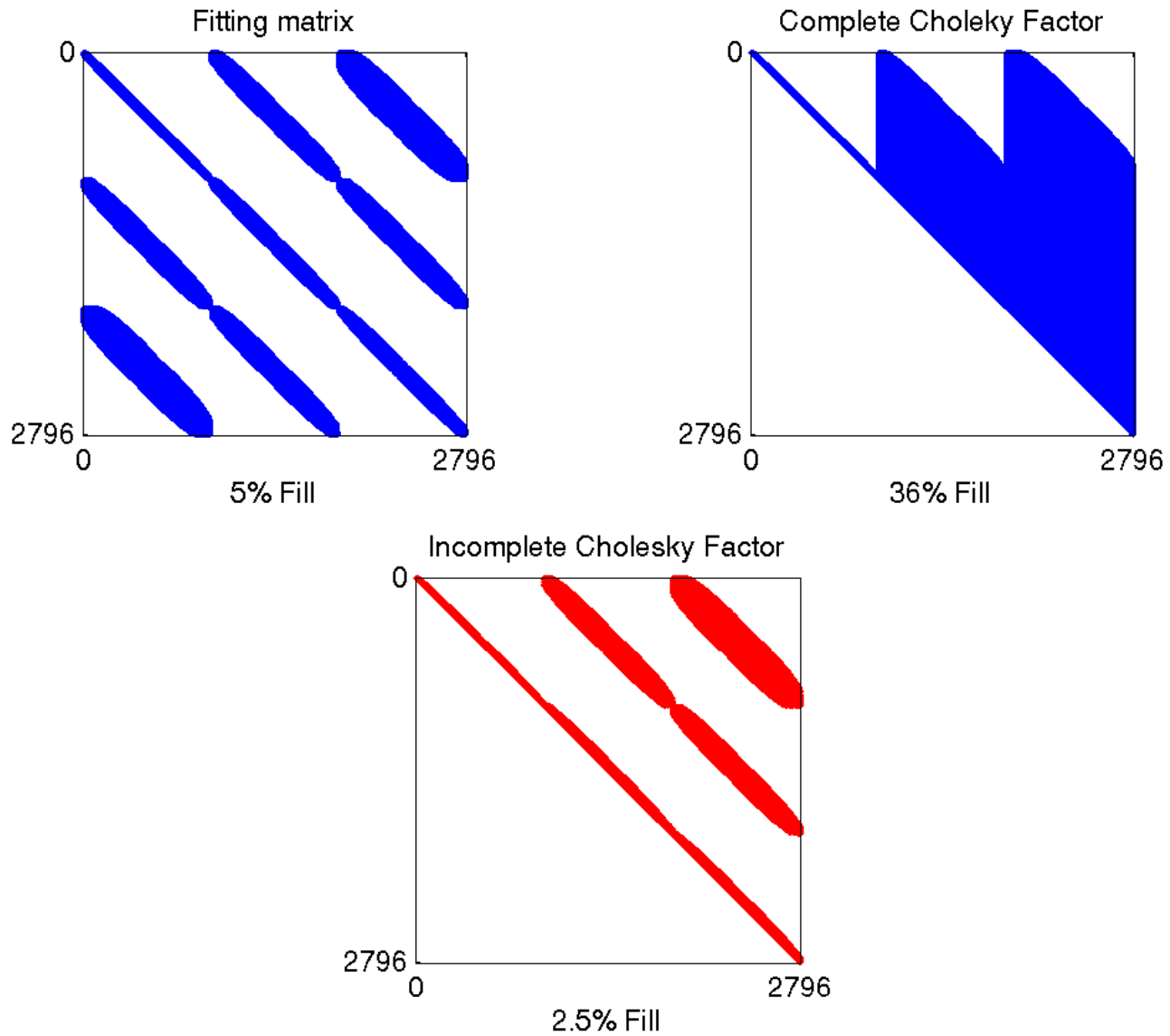


Fig.3

Estimation, 6-layer profile, 5 WFSs' using 5 NGS's. FoV diameter 100 arcsec, 1 V-cycle/CG iteration, 1 SGS iter/grid level, SNR = 20, $r_0 = 25\text{cm}$, $\Delta x = r_0$.

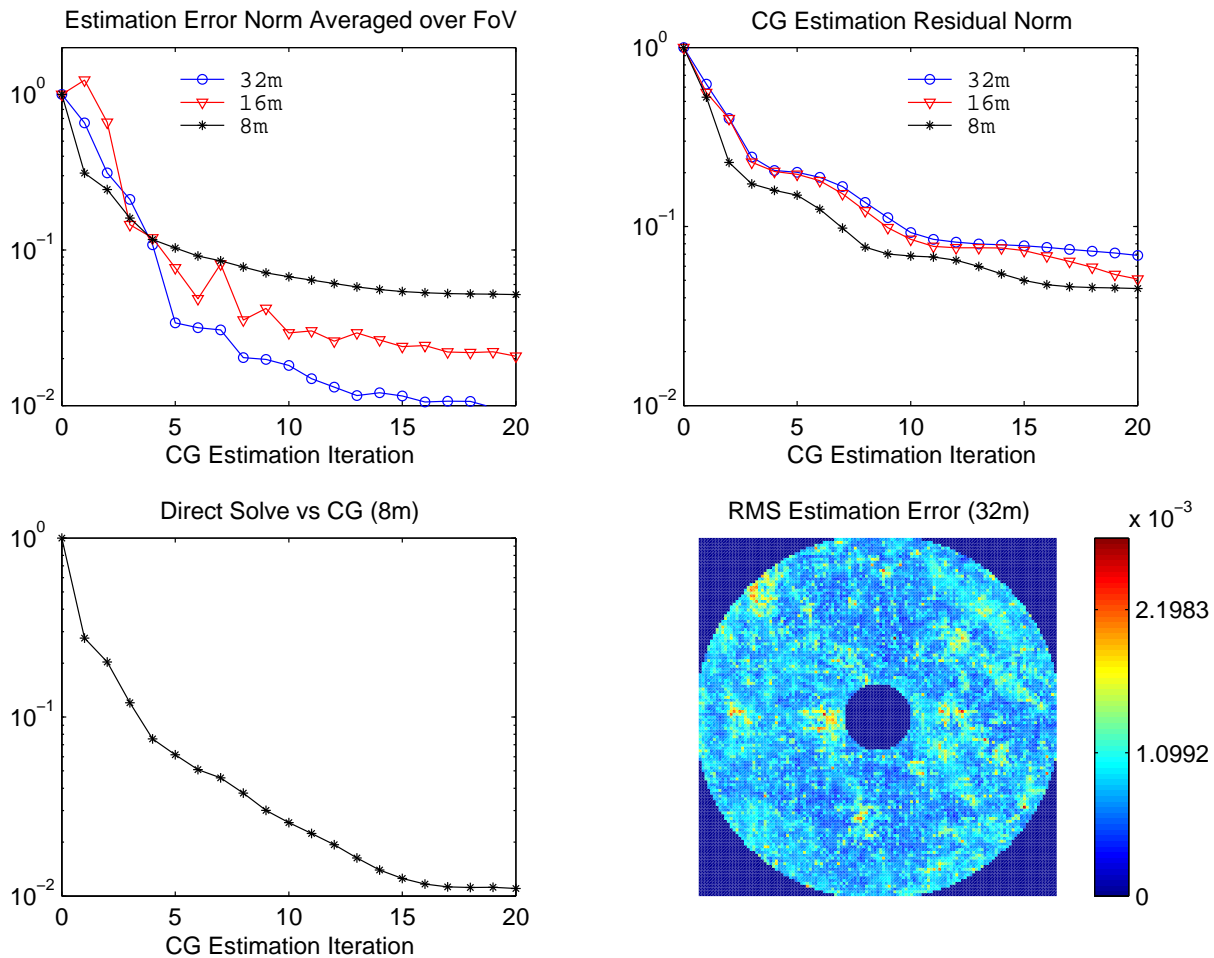


Fig.4

Fitting after 20 CG estimation iterations. Average over array of 5×5 observation directions. FoV diameter 100 arcsec. Incomplete Cholesky preconditioning, regularization parameter $\alpha = 1e-5$.

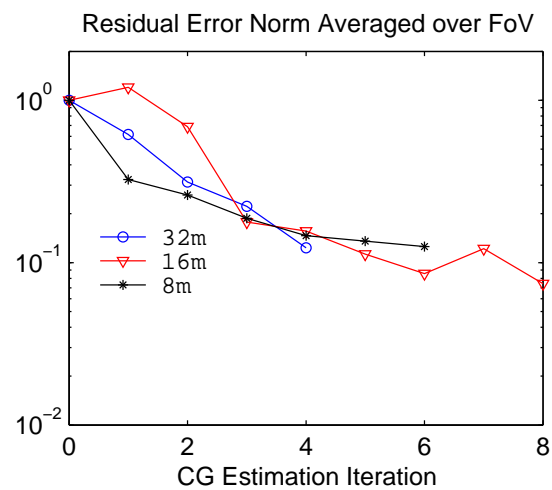
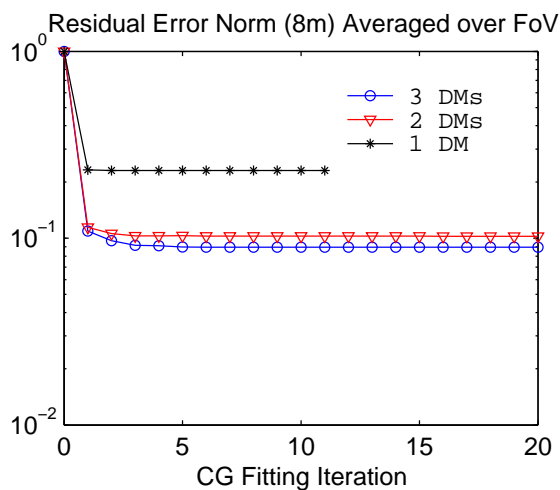
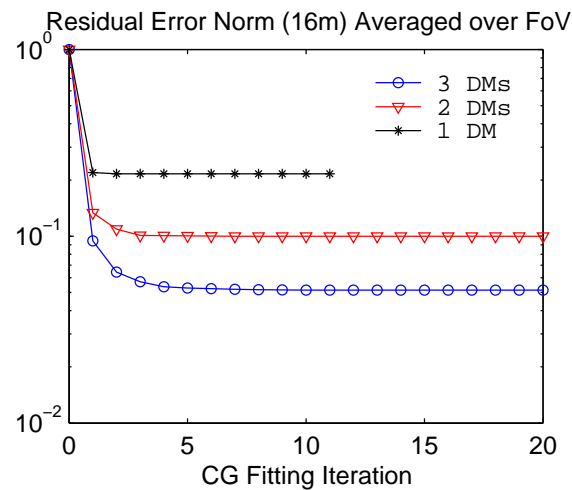
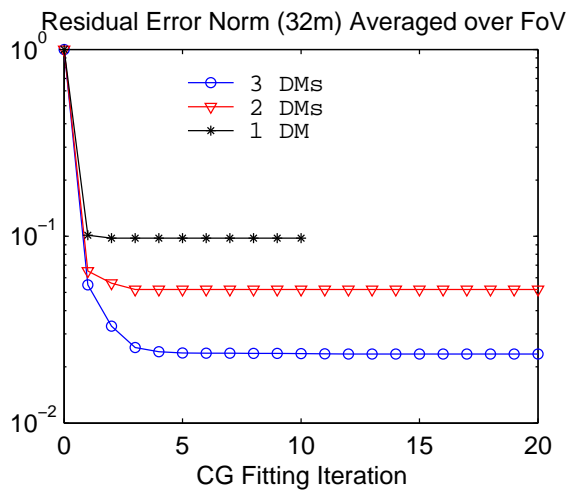


Fig.5

Fitting after 20 CG estimation iterations. Average over array of 5×5 observation directions. FoV diameter 100 arcsec. Incomplete Cholesky preconditioning, regularization parameter $\alpha = 1e-5$.

